

**Notation**

- $\mathbb{N}$ : the set of all positive integers.
- For all  $n \in \mathbb{N}$ ,  $[n]$  denotes the set  $\{1, 2, \dots, n\}$ .

**Q1.** Find the mistake in the proof provided for the claim given below.

**Claim:** All coins in a box have the same weight.

**Proof:** We will prove the above claim using induction.

*Base case.* If there is only one coin ( $n = 1$ ), then trivially, all coins in the box have the same weight.

*Inductive Hypothesis.* Assume that for any box containing  $n$  coins, all coins in the box have the same weight.

*Inductive Step.* Consider a box containing  $n + 1$  coins. Label them as  $C_1, C_2, \dots, C_{n+1}$ .

- Consider the first  $n$  coins:  $C_1, C_2, \dots, C_n$ . By the induction hypothesis, all of these coins have the same weight.
- Now consider the last  $n$  coins:  $C_2, C_3, \dots, C_{n+1}$ . Again, by the induction hypothesis, all of these coins have the same weight.
- Since  $C_1, C_2, \dots, C_n$  all have the same weight, and  $C_2, C_3, \dots, C_{n+1}$  all have the same weight, it follows that all  $n + 1$  coins must have the same weight.

*Conclusion.* By induction, we conclude that all coins in any box have the same weight.

[6]

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**Q2.** You need to compute the mean and the standard deviation of a list of  $N$  real numbers. **The value of  $N$  is not known in advance.** Fill in the blanks in the pseudocode given below, so that it prints the desired quantities.

**Pseudocode**

1. Initially, set variables  $A$ ,  $B$  and  $n$  to 0.
2. While there is at least one more input:
  - (a) Store the next input in  $x$ .
  - (b) Add \_\_\_\_\_ to  $n$ .
  - (c) Add \_\_\_\_\_ to  $A$ .
  - (d) Add \_\_\_\_\_ to  $B$ .
3. Print “*Mean* = \_\_\_\_\_”.
4. Print “*Standard Deviation* = \_\_\_\_\_”.

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**Q3.** A chessboard consists of 64 squares arranged in an  $8 \times 8$  grid, each containing positive integers. Two squares are *adjacent* if they share an edge.

The chessboard is called a *good chessboard* if the number on each square is the product of the numbers on the squares adjacent to it. Calculate the number of good chessboards.

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**Q4.** You have two stacks  $A$  and  $B$ . Your input is the following sequence of integers 1, 2, 3, 4, 5 (in that order). You are allowed to perform the following operations.

- S1: accept the next integer and **push** it in  $A$ .
- S2: **pop**  $A$  and push the output in  $B$ .

- S3: **pop**  $B$  and output the result.

The final output will be a permutation of the integers  $\{1, 2, 3, 4, 5\}$ . For example, the following sequence of operations

$S1, S2, S3, S1, S2, S3, S1, S1, S1, S2, S2, S2, S3, S3, S3$

would yield the following permutation:

1 2 3 4 5.

Check if the following two permutations can be generated by a sequence of operations as above. Give justification for your answer.

(a) 5 4 3 2 1

(b) 5 4 3 1 2

[3 + 3 = 6]

**Q5.** Given an array  $A$  (indexed starting from zero) of  $n \geq 2$  distinct integers, an element is a *local max* if it is strictly greater than its adjacent elements, i.e.,

- $A[0]$  is a local max if  $A[0] > A[1]$ .
- $A[n - 1]$  is a local max if  $A[n - 1] > A[n - 2]$ .
- If  $0 < i < n - 1$ , then  $A[i]$  is a local max if  $A[i] > A[i - 1]$  and  $A[i] > A[i + 1]$ .

Design an efficient algorithm to find a local max from the array  $A$ . Your algorithm should have time complexity  $O(\log n)$  to receive full credit.

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**Q6.** Let  $x$  and  $y$  be two non-negative  $n$ -bit integers. The operation  $x \& y$  represents the **bitwise AND** of  $x$  and  $y$ . Consider the following piece of code:

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1: Input: A non-negative integer  $x$ 
2:  $\ell \leftarrow 0$ 
3: while  $x \neq 0$  do
4:    $x \leftarrow x \& (x - 1)$ 
5:    $\ell \leftarrow \ell + 1$ 
6: end while
7: return  $\ell$ 
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What will be the output of the above piece of code for a given input  $x$ ? Explain your answer.

[6]

**Q7.** Consider 22 integers  $a_1, \dots, a_{22}$  satisfying the following properties

- For all  $1 \leq i \leq 22$ ,  $a_i \in \{-1, +1\}$ , and
- $|\{i : a_i = 1\}| - |\{j : a_j = -1\}| = 2$ .

Calculate the following

$$\sum_{S \subset [22] : |S|=10} \left( \prod_{i \in S} a_i \right).$$

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**Q8.** For  $n \in \mathbb{N}$  and  $k \in \{0, 1, \dots, 9\}$ , define

$$D(n) := \{i \in \mathbb{N} : i \text{ is an } n\text{-digit number}\}, \text{ and}$$
$$D(n, k) := \{j \in D(n) : j \text{ does not contain the digit } k\}.$$

Compute the following limit

$$\lim_{n \rightarrow \infty} \frac{\sum_{i \in D(n,k)} i^{-1}}{\sum_{j \in D(n)} j^{-1}}.$$

[6]

**Q9.** Let  $A$  be an  $n \times n$  matrix with real entries satisfying the following matrix equation:

$$A^2 = A - I_n,$$

where  $I_n$  is the  $n \times n$  identity matrix. Prove that the determinant of  $A$  is positive.

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**Q10.** Let  $A$  and  $B$  be two non-empty subsets of  $[n]$  with  $|A| = k$ ,  $|B| = \ell$  and  $|A \cap B| = m$ . Let  $R$  be a random subset of  $[n]$  where each element of  $[n]$  is independently included in  $R$  with probability  $1/2$  (and therefore not included in  $R$  with probability  $1/2$ ). Calculate  $\Pr[A \cap R \neq B \cap R]$ .

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