I. COMPUTER SCIENCE

C1. Let $A = [a_{ij}]$ be an $n \times n$ matrix with distinct integers as entries. The elements of the matrix may be read as usual, but only the following operations may be used to modify the matrix.

- SWAP_ROW(A, i, j) swaps row i of A with row j in $O(n)$ time;

- SWAP_COLUMN(A, i, j) swaps column i of A with column j in $O(n)$ time.

Note that modifying individual matrix elements is not allowed.

(a) Design an algorithm using the above operations to rearrange the matrix A in such a way that both the first row and the first column are sorted in ascending order.

(b) Analyze the time complexity of your algorithm.

[8+4=12]

C2. (a) Using *only* 2-input half-adders, implement NOT and 2-input OR gates. You can assume that logic 1 is available.

(b) Consider a circuit C that takes as input a valid month number (1 for January, 2 for February, ..., 12 for December) and outputs 1 if the month has 31 days, 0 for the other months. Assume that the month number is encoded using 4-bits $M_3 M_2 M_1 M_0$. Present a design of C using the minimum number of half-adders.

[(2+4)+6=12]

1

C3. (a) Consider a binary search tree T over integers. Let a, b, c be three given input integers such that a, b and c are known to be in T. Design an efficient algorithm to check if c is a common ancestor of a and b in T. You will get full credit only if your algorithm runs in $O(\text{height}(c))$ time. Here, $\text{height}(c)$ denotes the length of the path from the root to the node c.

(b) In a wireless sensor network deployed in a remote area, data packets are transmitted using an error correcting mechanism that employs a $(7, 4)$ Hamming code for error correction. If each data packet consists of 1400 bits, calculate the maximum number of bit errors that can be corrected in a single data packet transmission. Recall that Hamming$(7, 4)$ is a linear error correcting code that encodes four bits of data into seven bits by adding three parity bits.

$$[7+5=12]$$

C4. (a) To guess the cache block-size of the new generation processor on your desktop, you execute the following sequence of accesses to various *bytes* in your byte-addressable memory, starting with an empty cache.

$$0 \rightarrow 32 \rightarrow 128 \rightarrow 73 \rightarrow 8192 \rightarrow 255 \rightarrow 16384 \rightarrow 196$$

You observe that the cache hit rate is 0.5. You are told that the cache block-size is either 8, 16, 32, 64, or 128 bytes. Determine the possible cache block-size(s) with appropriate justification.

(b) Consider a non-pipelined processor with 500 MHz clock. The processor executes instructions sequentially one after another. A program, with the following instruction mix, needs 8 ms for its execution on this processor.

Instruction Type	Frequency	Clock Cycles
A	50%	4
B	30%	6
C	20%	1

What will be the execution time if only the count of type B instructions is reduced by half and the clock cycle time increases by 25% ?

[6+6=12]

C5. Let P_1, P_2, ..., P_{20} be a set of 20 cooperating processes, all of which have access to a shared integer X. For $i \in \{1, 2, 3, \ldots, 20\}$, process P_i calls do_something(i), the code for which is given below.

```
shared int X = 0;
void do_something(int i) { X += i; printf("%d %d   ", i, X); }
```

(a) What is the **minimum** possible value of X after all 20 processes complete execution? Justify your answer.

(b) Your task is to synchronise the execution of the above processes using semaphores so that the combined output of all 20 processes is as follows.

                1 1   2 3   3 6 ... 20 210

3

(i) Show how you would declare and initialise the necessary semaphore(s). The following examples show how semaphores can be declared and initialised.

```
shared semaphore S = 1;
shared semaphore mutex[m] = {0, 0, ..., 0};
```

(ii) Modify the body of the function `do_something(i)` by using appropriate `wait()` and `signal()` calls to synchronise P_1, P_2, ..., P_{20} so that the messages are printed in the aforesaid order.

[2+(2+8)=12]

C6. Consider designing a database management system (DBMS) for a small hospital. The said DBMS stores information of patients visiting the hospital on different dates for consulting one or more doctors on each day of visit. Each doctor specializes in diagnosing exactly one disease. On a visit, a doctor diagnoses the level ('None', 'Low', 'Moderate' or 'High') of a patient's disease. Assume that information on the following attributes need to be stored.

1. Patient ID (PID)
2. Name of patient (NoP)
3. Date of birth of patient (DoB)
4. Gender (G)
5. Address of patient (AoP)
6. Phone number of patient (PoP)
7. Date of visit of patient (DoV)

8. Disease diagnosed (DD)

9. Level of disease (LoD)

10. Doctor ID (DID)

11. Name of doctor (NoD)

12. Specialization of doctor (SoD)

On the day of the first visit, each patient is assigned a unique PID based on the NoP and PoP. A patient may have one or more address(es), but only a single phone number. Note that multiple patients may be registered with the same phone number.

(a) Write down all possible functional dependencies that should be considered for proper design of a relational model for the above problem.

(b) Assume that the entire information is maintained in a single relational schema R. Decompose R into several relational schemas so that

- Decomposition is lossless; AND
- Redundancy is minimized as far as possible; AND
- Dependency is preserved as far as possible.

[5+7=12]

II. MATHEMATICS FOR COMPUTER SCIENCE

Notation

$\mathbb{N}$: the set of natural numbers; $\mathbb{R}$: the set of real numbers.

MC1. (a) Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} \frac{x^3 y}{x^6 + y^2} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Is the function continuous at the point $(0,0)$? Justify your answer.

(b) Let $p(x)$ be a polynomial of degree n with real coefficients and $p'(x)$ denote the first derivative of $p(x)$. Examine the following graph of $p(x)$ in the interval $(-3,4)$ and answer the questions below with justifications.
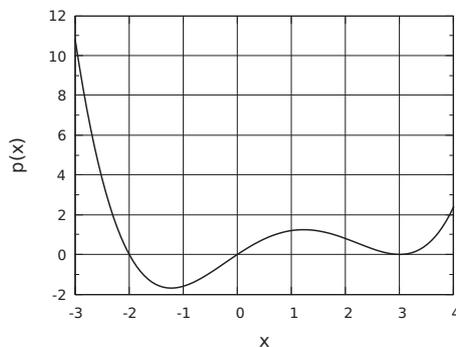
(i) What are the real roots of $p(x)$ in the interval $(-3,4)$?

(ii) What is the minimum possible value of n?

(iii) How many distinct real roots does the polynomial $p'(x)$ have in the interval $(-3,4)$?

$$[7+(1+2+2)=12]$$

MC2. (a) Let V be a vector space, T a linear transformation on V, and T^2 denote the composite $T \circ T$. Prove that if $\ker\ T = \ker\ T^2$, then

$$\ker\ T \cap \mathrm{img}\ T = (0)$$

where ker denotes the kernel (i.e., null space) and img denotes the image.

(b) Let A be an $n \times n$ real symmetric matrix, where $n > 1$. Show that
$$U = A^4 + A^3 + I$$

cannot be singular.

[6+6=12]

MC3. Let $X(K, u)$ be a random variable with probability mass function defined as

$$f_{K,u}(x) = \begin{cases} \frac{K}{x^u} & \text{for } x \geq 5, \quad x \in \mathbb{N} \\ 0 & \text{otherwise,} \end{cases}$$

where $K > 0$ and $u \geq 2$.

(a) Show that K is an increasing function of u.

(b) Prove that

$$P(X(K_1, u) > t) > P(X(K_2, u) > t), \ \forall t > 5$$

if and only if $K_1 < K_2$.

[4+8=12]

MC4. (a) Let t_n be the number of natural numbers, which are co-prime with 35 and less than 10^n. Determine t_n.

(b) Let p, q, r be three integers such that

$$p^2 + q^2 = r^2.$$

Show that 60 divides pqr.

[6+6=12]

MC5. (a) Let G be a directed graph, and u, v be any two distinct vertices in G. Show that if G is acyclic, then it is always possible to add at least one of (u, v) or (v, u) to G without creating a cycle.

(b) A "tournament" is a directed graph $G = (V, E)$ in which for each pair of distinct vertices $u, v \in V$, exactly one of (u, v) or (v, u) belongs to E. Let G be a tournament on n vertices, where $n > 3$. For any $v \in V$, $I(v)$ and $O(v)$ denote the set of vertices such that each vertex in $I(v)$ has a directed edge towards v, and each vertex in $O(v)$ has a directed edge from v, respectively.

(i) If every edge of G is contained in a directed triangle (directed cycle of length 3), show that each vertex of $O(v)$ has a directed edge towards some vertex of $I(v)$ and vice versa.

(ii) Hence, or otherwise, prove that there is a directed cycle of length 4 in G.

[4+(4+4)=12]

MC6. (a) Consider the pushdown automata P given in the figure below. Its input alphabet is $\Sigma = \{a, b\}$, and the stack alphabet is $\Gamma = \{a, b, z_0\}$, where z_0 is the stack start symbol. A label $a, X/\alpha$ of an arc represents that for the input symbol a, the top symbol X of the stack will be replaced by the string α in reverse order. The symbol ϵ denotes the null string.
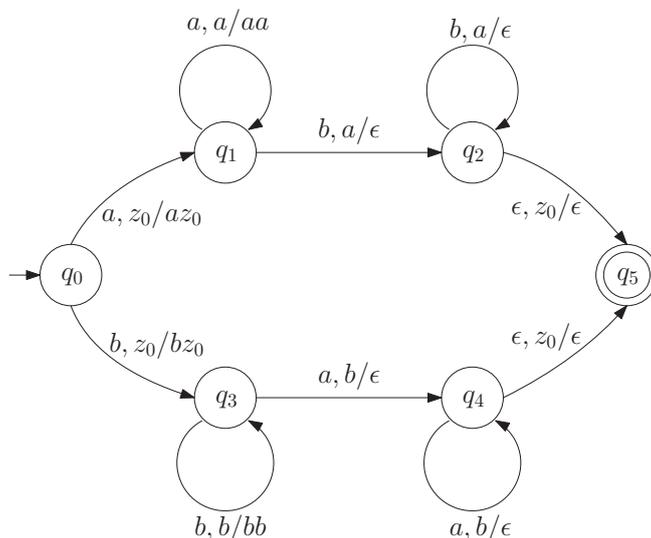
Assume that P accepts a language L by empty stack. Identify L and write a context-free grammar for L.

(b) Let L_1 be a regular language and L_2 be a non-regular language over an alphabet Σ. State the conditions, with justifications, under which

  (i) $L_1 \cup L_2$ is non-regular;

  (ii) $L_1 \cup L_2$ is regular.

$$[(2+4)+(3+3)=12]$$