A1. If α, β, γ are roots of $x^3 + 6x + 1 = 0$, then find the value of

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}.$$

A2. Let $X = \{0, 1, 2, 3\}$, $F = \{1893, 1972\}$, and $G = \{7, 13, 29\}$. Consider the following two functions

$$f : X \to F, \qquad \text{and} \qquad g : X \to G.$$

Determine the number of possible functions $h(x)$, where

$$h(x) = f(x) \times g(x), \quad x \in X.$$

A3. Consider the definite integral

$$I = \int_1^3 \frac{dx}{1 + x^2}.$$

Show that I lies between $\frac{3}{10}$ and $\ln 2$.

A4. A queue, Q, contains the integers $1, 2, \cdots, 2m$, in that order, with 1 being at the head. You are also given an empty stack S. Write pseudocode to show how you would rearrange the elements of Q in the following order:

$1, 2m, 3, 2m-2, 5, 2m-4, \cdots, 6, 2m-3, 4, 2m-1, 2$

using the operations below.

- `ENQUEUE(i, Q)`
- `x = DEQUEUE(Q)`
- `PUSH(j, S)`
- `y = POP(S)`

A5. Consider the function definition given below:

```
int isiJRFTest (int a, int b, int c, int d)
{
   if (b == a)
       return c * (a - 1) + 1;

   return isiJRFTest (a, b + 1, c + d, d * a);
}
```

What does isiJRFTest (2024, 0, 0, 1) return? Justify your answer. Assume that your computer can store and manipulate as large integers as necessary without any error.

A6. In a 100 meter running competition, 25 students are competing for 3 prizes. There are 5 running tracks, each of length 100 meter, and thus a maximum of 5 students can take part in a single race. You are allowed to arrange as many races as you wish to determine the winners of the 3 prizes.

Consider the following assumptions.

- If student A defeats student B in a race, A can be concluded to be faster than B across all races.
- The students run at distinct speeds, so that no race ends up in a tie.
- There is no way to measure the speeds of the students from a race.

Design an arrangement of 7 races, which determines the top 3 fastest students.

A7. Let A be a 10×10 matrix such that each column of A represents a permutation of the numbers in the set $B = \{2, 5, 8, \cdots, 26, 29\}$. Show that 155 is an eigenvalue of A.

A8. Recall that a single variable polynomial in x is of the form

$$f = \sum_{i=0}^{n} a_i x^i$$

where all a_is are integers, and the degree of f be the maximum i such that a_i is non-zero. Let poly be a data structure that stores a polynomial and int be the integer data type. The following operations you are allowed to perform on poly.

- poly create_poly(int n): returns a zero polynomial, that is, $f = \sum_{i=0}^{n} a_i x^i$ with all a_is set to zero,
- int get_coeff(poly f, int i): returns the coefficient a_i of x^i of f,
- int set_coeff(poly f, int a, int i): sets the coefficient of x^i of f to a,

Only $+, -, *$ are the allowed operations on the data structure int with the usual meanings. You can use the following for looping.

- for i = j to k: represents a for loop indexed at i with the start value as j and the end value as k with unit increment.

For a given $f = \sum_{i=0}^{n} a_i x^i$, we define $f^2 = \sum_{i=0}^{n} a_i^2 x^{2i}$. Your task is to write pseudocode for computing f^2 given f and its degree n with the help of the above allowed operations.

A9. Consider a reservoir fitted with a set of taps and drains. The partial pseudocode given below takes as input - (i) the capacity of the reservoir, and (ii) details about the taps and drains attached to the reservoir. Fill in the blanks below, so that the pseudocode produces the correct output.

Input:

L	capacity of the reservoir in litres
T	number of taps attached to the reservoir
D	number of drains attached to the reservoir
t_i	time in minutes that it takes to fill an initially empty reservoir if all drains are stopped, and only the i-th tap is opened ($1 \leq i \leq T$)
d_j	time in minutes that it takes to drain an initially full reservoir if all taps are shut off, and only the j-th drain is opened ($1 \leq j \leq D$)

Pseudocode

I Initially, set variables X and Y to 0.

II **for** i in $\{1, 2, 3, \ldots, T\}$
$X \longleftarrow X+$ _____________.

III **for** j in $\{1, 2, 3, \ldots, D\}$
$Y \longleftarrow Y+$ _____________.

IV **If** X _____ Y
then print "*A partially empty reservoir never fills up if all*

taps and drains are kept open."
else print "*An initially empty reservoir fills up in* ________ *minutes if all taps and drains are kept open.*"

V **If** X _____ Y
then print "*A partially full reservoir never becomes empty if all taps and drains are kept open.*"
else print "*An initially full reservoir becomes empty in* _____ *minutes if all taps and drains are kept open.*"

A10. Suppose a fair die is rolled 7 times independently. What is the probability that at least one of the six sides of the die never shows up in these 7 rolls?