## I. COMPUTER SCIENCE

C1. (a) A synchronous sequential circuit takes a stream of input bits and produces the output $y = 1$ whenever the sequence 111 occurs, and 0 otherwise. Note that the input stream may contain overlapping instances of the sequence 111.
For example, if the input is 01011111...,
the corresponding output is 00000111....

Design such a circuit using $D$ flip-flops, and draw the corresponding logic diagram. [3+2]

(b) A binary display $\mathcal{C}(N : 2^N - 1)$ takes an $N$-bit input which represents an integer $k$ $(0 \leq k \leq 2^N - 1)$, and displays $2^N - 1$ bits as output, of which the $k$ least significant bits are 1, and the remaining bits are 0.

For example, with $N = 3$ and $k = 6$, i.e., if the input is 110, the display output is 0111111.

Design a binary display $\mathcal{C}(3 : 7)$ using the minimum number of 2-input AND, OR and NOT logic gates. Give the corresponding Boolean expression for each output bit. [4+3]

C2. (a) Consider the two instructions $SLQ\ (X, Y, Z)$ and $ST\ (X, v)$, defined as follows:

$$\begin{aligned} ST\ (X, v) &\ :\ Mem[X] = v \\ SLQ\ (X, Y, L) &\ :\ Mem[Y] = Mem[Y] - Mem[X] \\ &\quad\ \ if\ (Mem[Y] \leq 0)\ goto\ L \end{aligned}$$

$X, Y$ are memory locations, $L$ is a symbolic label and $v$ is an integer. Using *only* the $ST$ and $SLQ$ instructions described above, realise the instruction $ADD\ X, Y$ defined as:

$$ADD\ (X, Y) \quad : \quad Mem[Y] = Mem[Y] + Mem[X]$$

Assume a sequential instruction execution model, where instructions are executed one after another in the order they are specified, unless control is explicitly transferred to a different label (as in *goto L* above). In your answer, clearly assign labels to instructions as appropriate. [6]

(b) Consider a balanced binary search tree in which each node has an additional field SIZE which is the number of nodes contained in the subtree rooted at this node. Thus, the SIZE field of a leaf is 1, and that of the root is $n$, the total number of nodes in the tree. Design a $O(\log n)$ procedure to find the $k^{th}$ $(1 \leq k \leq n)$ smallest element in the search tree. [6]

C3. Consider a hypothetical system with the following specifications:

- an instruction takes 0.5 ns of CPU time on average and has two memory references for the operands;
- the processor uses a two-level paging scheme for virtual to physical address translation;
- both page tables are in main memory;
- both the virtual address and the physical address are of 32-bits;
- for address translation, the 10 most significant bits of the virtual address are used for indexing of the first page table, the next 10 bits are used for indexing of the second page table, and the last 12 bits are for offset within a page;

- the processor has a TLB (used during address translation) with a hit ratio of 96% and a cache with a hit ratio of 90%;
- main memory access time is 10 ns, cache access time is 1 ns, and TLB access time is 1 ns;
- servicing a page fault takes 1 ms.

If page faults occur at a rate of one in every $10^4$ instructions, compute the

(a) effective address translation time, [4]

(b) effective memory access time, and [4]

(c) effective instruction execution time. [4]

C4. Assume that two computers $A$ and $B$ are connected through a router $R$. The IP address of $A$ is 204.198.65.198, while that of $B$ is 204.198.3.61. On an average, the router receives 990 packets per second following a Poisson distribution and transmits 1000 packets per second following an exponential distribution. The router has only one processor and sufficient memory to store incoming packets.

(a) Determine

(i) how many times a packet has to go through the network layer and the data link layer when $A$ transmits data to $B$;

(ii) the average number of packets in the router at steady state. [4 + 4]

(b) Explain if $A$ and $B$ can transmit data to each other properly when $A$ has subnet mask 255.255.192.0 and $B$ has subnet mask 255.255.128.0. [4]

C5. Consider the following relational schema containing information about a distributor's stock of medicines.

*Medicines* ( *Name*, *Manufacturer*, *UnitPrice*, *FactoryLocation*, *BatchNo*, *MfgDate*, *ExpDate*, *ShopID* )

The attributes above have the obvious interpretations, i.e., each record stores the *Name* of a medicine, its *Manufacturer*, *UnitPrice*, the location of the factory where it was manufactured, the corresponding batch number, manufacturing and expiry dates, and the identifier of a shop that needs the medicine. Consider the following instance of the *Medicines* schema.

| Name | Manu-facturer | Unit Price | Factory Loca-tion | Batch No | Mfg Date | Exp Date | Shop ID |
|---|---|---|---|---|---|---|---|
| Coldgone | National Drugs | 10 | Mumbai | 42 | 12/21 | 05/23 | 87 |
| Allvits | National Drugs | 40 | Chennai | 33 | 09/21 | 09/24 | 62 |
| Allvits | National Drugs | 40 | Chennai | 33 | 09/21 | 09/24 | 56 |
| Zeropain | WB Pharma | 30 | Chennai | 39 | 11/21 | 05/23 | 47 |
| Antiprez | RT&Co. | 10 | Kolkata | 57 | 01/22 | 01/24 | 99 |
| Vitcee | WB Pharma | 10 | Kolkata | 77 | 01/22 | 06/23 | 98 |
| Vitcee | Doctor's | 12 | Kolkata | 78 | 02/22 | 06/22 | 98 |

(a) Determine the functional dependencies that hold on the above table, and are of the form $X \rightarrow Y$ where $X$ and $Y$ are single attributes of the corresponding schema. [6]

(b) Using the above functional dependencies, find the possible key(s). [2]

(c) Decompose the above schema into the minimum number of BCNF schemas. [4]

C6. (a) Let $A$ and $B$ be two unsorted arrays, each having $n$ distinct integers. Define

$$Pred(a) = \max_{b \in B}\{b < a\}, \quad a \in A,$$

i.e., for $a \in A$, $Pred(a)$ is the largest integer $b \in B$ such that $b < a$. Present an $O(n \log n)$-time algorithm for computing $Pred(a)$ for all $a \in A$, and justify the running time of your algorithm. [6]

(b) Show that the above problem can be solved in $O(n)$ time if the input arrays $A$ and $B$ are sorted. [6]

## II. MATHEMATICS FOR COMPUTER SCIENCE

MC1. For any positive integer $n$, let $d(n)$ denote the number of positive divisors of $n$.

(a) Suppose $n = p_1^{a_1} \dots p_k^{a_k}$ where $p_i$s are distinct prime numbers and $a_i \geq 1$. Show that

$$d(n) = (a_1 + 1)(a_2 + 1) \dots (a_k + 1).$$

[4]

(b) Suppose $s > t > 0$ are two integers. Show that

$$(s - t) \text{ divides } [d(n^s) - d(n^t)].$$

[4]

(c) Suppose $m$ is another positive integer. Show that

$$s \text{ divides } [d(n^s) - d(m^s)].$$

[4]

MC2. (a) A fair coin is tossed 50 times. What is the probability of obtaining a head in the first toss, given that exactly 30 heads were obtained in 50 tosses? [6]

(b) (i) Show that, for every non-negative integer $n$, the number $(\sqrt{2}+\sqrt{3})^{2n}$ is of the form $(a_n+b_n\sqrt{6})$, where $a_n$ and $b_n$ are also integers. [2]

(ii) Write simultaneous recurrence relations for $a_n$ and $b_n$ and the relevant boundary conditions. [4]

MC3. A *vertex colouring* of a graph $G(V, E)$ is an assignment of colours to $V$ such that no two adjacent vertices are assigned the same colour. Let the graph $G$ be an odd cycle with $n$ vertices.

(a) If $n = 7$, then construct a vertex colouring of $G$ such that in any independent set of size 3 in $G$, at least two vertices have the same colour. [6]

(b) Prove that, if $n \geq 9$, then in any vertex colouring of $G$, there must be an independent set of size 3 in which each vertex has a distinct colour. [6]

MC4. (a) Given a finite alphabet $\Sigma$, show that the number of regular languages over $\Sigma$ is countable. [6]

(b) Show that the language $L = \{a^j b^k \mid j \geq k^2\}$ is not context-free. [6]

## III. MATHEMATICS

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**Notation**

$\mathbb{R}$ = the set of real numbers

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M1. (a) Let $P_1, P_2$ be two projection matrices, i.e., $P_1^2 = P_1$ and $P_2^2 = P_2$. Let $\mathbf{x}$ be an eigenvector of $P_1 + P_2$, and let $\mathbf{y}$ be any vector in $\text{span}\{\mathbf{x}, P_1\mathbf{x}\}$. Prove that $P_1\mathbf{y}$ and $P_2\mathbf{y}$ both belong to $\text{span}\{\mathbf{x}, P_1\mathbf{x}\}$. [7]

(b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $T^2 = 0$. Prove that the rank of $T$, i.e. $\dim(Im(T))$, cannot exceed $n/2$. [5]

M2. Let $k$ be a field and $R$ the ring $k[X]/(X^2)$.

(a) Prove that $R$ has *exactly one* prime ideal, and describe the prime ideal. [7]

(b) Examine whether $R$ is isomorphic as a ring to the product ring $k \times k$ (with coordinate wise addition and multiplication). [5]

M3. Let $S_m$ denote the finite symmetric group defined over a finite set of $m$ symbols, consisting of the permutations that can be performed on the $m$ symbols.

A function $\phi : \{0,1\}^m \to \{0,1\}$ is said to be transitive if there exists a group $G \subseteq S_m$ with at least 2 elements, such that for all $\sigma \in G$ and for $1 \leq i \leq m$, $x_i \in \{0,1\}$

$$\phi(x_1, \ldots, x_m) = \phi(x_{\sigma(1)}, \ldots, x_{\sigma(m)}).$$

Prove that if $f : \{0,1\}^n \to \{0,1\}$ is transitive and $g : \{0,1\}^k \to \{0,1\}$ is transitive, then the function $(f \circ g)$ is also transitive.

Here, $(f \circ g)$ is the function from $\{0,1\}^{nk} \to \{0,1\}$ defined as

$$\begin{aligned}&(f \circ g)(x_{1,1}, \ldots, x_{1,k}, x_{2,1}, \ldots, x_{2,k}, \ldots, x_{n,1}, \ldots, x_{n,k}) \\ &\quad = f(g(x_{1,1}, \ldots, x_{1,k}), g(x_{2,1}, \ldots, x_{2,k}), \ldots, g(x_{n,1}, \ldots, x_{n,k})).\end{aligned}$$

[12]

M4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be convex if for all $\lambda \in [0,1]$ and $x, y \in \mathbb{R}$,

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y).$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. For distinct $x, y, z$, define

$$\Phi_f(x,y,z) := \frac{\left(\frac{f(x)-f(z)}{x-z} - \frac{f(y)-f(z)}{y-z}\right)}{(x-y)}.$$

Prove that $f$ is convex if and only if $\Phi_f \ge 0$. [12]

M5. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. For any non-empty interval $I \subset \mathbb{R}$, let $U(f,I)$ and $L(f,I)$ denote the least upper bound (lub) and the greatest lower bound (glb) of $\{f(x) : x \in I\}$, respectively, and let

$$w(f,I) = U(f,I) - L(f,I).$$

Note that $w(f,I) \ge 0$.

For any $x \in \mathbb{R}$, let

$$w(f,x) = \text{glb}\{w(f,I) \mid I \text{ is an open interval containing } x\}.$$

Prove that the function $f$ is continuous at $x$ if and only if $w(f,x) = 0$. [12]

M6. (a) Let $X$ be a Hausdorff space. Prove that any two disjoint compact subsets of $X$ can be separated by disjoint open sets. [4]

(b) Let $X$ and $Y$ be topological spaces and $f : X \to Y$ a surjective continuous map. Assume that $X$ is Hausdorff, $f$ is a closed map and $f^{-1}(y)$ is compact for every $y \in Y$. Prove that the space $Y$ is Hausdorff. [8]