Notation

$\mathbb{Z}$ = the set of integers

$\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$

$\mathbb{R}$ = the set of real numbers

$\mathbb{C}$ = the set of complex numbers

A1. Let p and q be two primes such that $p \neq q$ and 6 divides $(p+q)$. Prove that 6 does not divide (p^2+q^2). [4]

A2. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = x^5 - 3x^3 - 2x^2 + 10x - 7$$

is bijective. [4]

A3. Six numbers are chosen at random from the set $\{1, 2, \ldots, 13\}$ without replacement.

(a) Calculate the probability that their sum is even. [5]

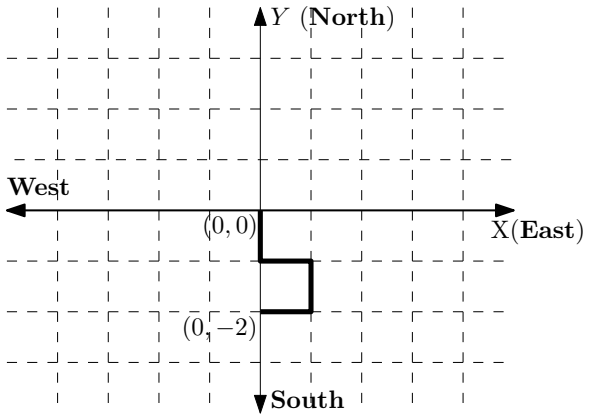
(b) Compute the probability that none of the six numbers is even, given that the sum is even. [3]

A4. At a party, you realise that everyone, except you, has shaken hands with exactly three other people; you have shaken hands with only one other person. Determine the minimum number of attendees at this party. [6]

A5. Let G, G_1, G_2 be groups such that $G \subseteq G_1 \cup G_2$. Prove that either $G \subseteq G_1$ or $G \subseteq G_2$. [5]

A6. A person, located at (0,0) in the X-Y plane, starts moving in steps.

(a) At each step, the person chooses one of the four directions, North, South, East or West, with equal probability, and moves by one unit in the corresponding direction. The figure below shows a possible path covered by the person in 4 steps.



Fill in the body of the for-loop in the pseudocode below to simulate the random movement of the person for n steps, and report the final coordinates of the person. In your code, you may use a function *toss_coin*() wherever required. This function simulates a fair coin being tossed once, and randomly returns either HEAD or TAIL. [5]

Input: n
Initialise current-x-coordinate = 0,
current-y-coordinate = 0.
for i in $1 \ldots n$ **do**
**** Write the code for this part in your answer booklet. ****
end for
Report current-x-coordinate, current-y-coordinate.

(b) Modify your code to handle a situation in which the person chooses, at each step, one of the **five** following options with equal probability: stay at the same location, or move by one unit towards the North, South, East or West.

If you clearly mark which parts of your code will remain the same, you need not re-write these unchanged parts in your answer to (b). [5]

A7. Let I_n be the $n \times n$ identity matrix, and let A be an $n \times n$ matrix such that $A^m = 0$ for some positive integer m. Show that I_n - A is an invertible matrix by explicitly expressing I_n - A as a polynomial in A. [4]

A8. Consider the sentence **S** below.

$$\textbf{S} : \text{There exist } N_0 \in \mathbb{N} \text{ and } c \in \mathbb{R} \text{ such that for all } n \geq N_0$$
$$f(n) \leq c \cdot (g(n))^2 \text{ and } g(n) \leq c \cdot (f(n))^2.$$

Consider the negation of **S**, and hence prove that if $f(n) = n^2$ and $g(n) = 2^n$, then **S** is not true for $f(n)$ and $g(n)$. [6]

A9. Consider the following polynomial:

$$f(x) = x^6 + 3x^5 + 5x^4 - 7x^3 + 9x^2 + 11x + 1.$$

Let $\alpha_1, \ldots, \alpha_6 \in \mathbb{C}$ be the roots of the polynomial $f(x)$.

(a) Compute $\displaystyle\sum_{i=1}^{6} \alpha_i$ and $\displaystyle\sum_{1 \leq i < j \leq 6} (\alpha_i \cdot \alpha_j)$. [2]

(b) Prove that not all the roots of the polynomial $f(x)$ are real.

[HINT: You may use (a).] [2]

(c) Deduce that the following polynomial also has non-real roots:

$$g(x) = x^6 + 11x^5 + 9x^4 - 7x^3 + 5x^2 + 3x + 1.$$

[3]

A10. A 14 ft. by 30 ft. room is 49 ft. high. There is a lizard on one of the smaller walls. It is equidistant from the two vertical edges of the wall, and 8 ft. below the ceiling. The lizard wants to capture a stationary cockroach. The cockroach is on the opposite wall, equidistant from the two vertical edges of the wall and 8 ft. above the floor. The lizard can only walk on a surface, but cannot jump. Compute the length of the shortest path the lizard can take. [6]