Part I: Section M (Common for Both Streams)

(Answer any **TWO** questions.)

1. (a) Find the minimum value of $f(x, y)$, using only the fact that $AM \geq GM$ for two positive numbers, where

$$f(x, y) = 4x^2 + \frac{x}{y^2} + \frac{4y}{x},\ x, y > 0.$$

Note: No credit will be given if results from calculus are used.

(b) Without expanding show that the value of the matrix D is real, where

$$D = \begin{pmatrix} 2 & 3+i & -1 \\ 3-i & 0 & -1+i \\ -1 & -1-i & 1 \end{pmatrix}.$$

(c) If a_1, a_2, a_3 and a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, then show that

$$\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{2a_2}{a_2 + a_3}.$$

[8+5+7 = 20]

2. (a) Suppose A is a 3×3 matrix, where

$$A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

i. Obtain the eigen values and eigen vectors of A.

ii. Is it possible to obtain three linearly independent eigen vectors for A? Justify your answer.

iii. Check whether A can be diagonalized as $P^{-1}AP$, where P is the modal matrix of A.

(b) If y_1 and y_2 be the solutions of the equation

$$\frac{dy}{dx} + Py = Q,$$

where P and Q are functions of only x and $y_2 = y_1 z$, then show that

$$z = 1 + ae^{-\int \frac{Q}{y_1}\, dx}.$$

[(4+3+4)+9 = 20]

3. (a) Consider the following two functions:

$$\begin{aligned} f(x) &= x^3 + Ax^2 + x + B \\ g(x) &= x^3 + x^2 + Bx + A \end{aligned}$$

where, A and B are integers and B is not even.

Suppose the following two conditions hold:

i. $f(1) = 0$.

ii. The roots of $g(x) = 0$ are squares of the roots of $f(x) = 0$.

Find the value of $A^{2025} - B^{2025}$.

(b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x$, $\forall x \in R$. Then find $i)$ fog and $ii)$ gof.

(c) Two parallel chords of a circle of radius 2 are at a distance $(1 + \sqrt{3})$ apart. If the two chords subtend angles $\frac{\pi}{k}$ and $\frac{\pi}{2k}$, $k > 0$, at the centre, then find the value of $[k]$.

[7+($2\frac{1}{2} + 2\frac{1}{2}$)+8 = 20]

4. (a) Suppose AX, BY and CZ are three concurrent cevians other than altitudes, medians or angle bisectors of the triangle ABC given in Figure 1. (Cevian: a line segment that connects a vertex of a triangle to a point on the opposite side)

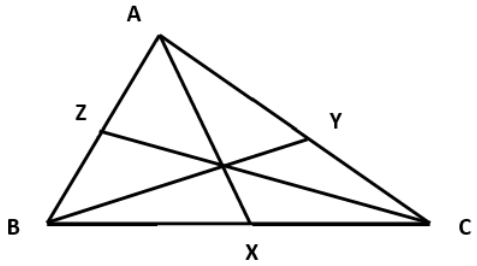

Figure 1

Show that $\frac{BX}{XC} \times \frac{CY}{YA} \times \frac{AZ}{ZB} = 1$.

(b) Use de Moivre's thorem to express $\sin^4\theta \cos^4\theta$ in terms of *sines* and *cosines* of multiples of θ.

[12+8 = 20]

Part II: Section S (Statistics Stream)

(Answer any **FOUR** questions.)

5. (a) A game host puts three balls randomly into three identical boxes. Each box may contain 0, 1, 2 or 3 balls. The player chooses any two boxes to open and if all the three balls are found in these two chosen boxes, then the player wins the game. Find the probability that the player wins the game.

   (b) A pair of unbiased dice is thrown until the sum of two faces is either six or seven. Find the probability that the sum seven occurs first.

   (c) If a fair coin is tossed an even number $2n$ times, show that the probability of obtaining more heads than tail is

$$\frac{1}{2}\left(1-\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right).$$

[7+7+6 = 20]

6. (a) Suppose that the number of customers visiting a restaurant on a given day is N, with $N \sim \text{Poisson}(\lambda)$. Assume that each customer purchases a drink with probability p. Furthermore, purchase of one customer is independent of the purchase of other customers and independent of N.

   Let X be the number of customers who purchase the drink and Y be the number of customers who do not purchase the drink.

   i. Find the marginal probability mass function of X and Y.

   ii. Find the joint distribution of X and Y.

   iii. Are they independent?

(b) A population consists of three villages V_1, V_2 and V_3 with average farm size x_1, x_2 and x_3 respectively. A probability sample consisting of two villages is selected to estimate the population average μ ($\mu = \frac{x_1+x_2+x_3}{3}$). A consultant suggests two estimators T_1 and T_2 as depicted below:

Sample S	**Prob (S)**	**Estimator T_1**	**Estimator T_2**
V_1, V_2	$\frac{1}{3}$	$\frac{x_1+x_2}{2}$	$\frac{x_1+x_2}{2}$
V_2, V_3	$\frac{1}{3}$	$\frac{x_2}{2} + \frac{x_3}{3}$	$\frac{x_2+x_3}{2}$
V_1, V_3	$\frac{1}{3}$	$\frac{x_1}{2} + \frac{2x_3}{3}$	$\frac{x_1+x_3}{2}$

Find the condition(s) under which the estimator T_1 would be preferable.

[(5+5+2)+8 = 20]

7. (a) A company manufactures a certain type of rechargeable battery and the lifetime of each battery under stress (in days) is assumed to follow an exponential distribution with mean θ. Suppose there are 3 types of batches of rechargeable batteries, viz., "*good*", "*medium*"and "*bad*", having average lifetime under stress as 4 days, 1 day and 1/4 day respectively. Five rechargeable batteries are selected from a batch at random and their lifespan under stress turns out to be 4 days, 2 days, 1 day, 2 days and 1 day (in coded units). Based on the above data, can you find an MLE of batch-type (i.e., whether it is "*good*", "*medium*"or "*bad*")?

[Assume $\log 3 \approx 1.1$ and $\log 2 \approx 0.7$]

(b) A marathon run is conducted where the total distance is

θ km. Five runners participated and the lengths (in km) covered by them are as follows:

8, 5, 9, 6, 10.

i. Assuming the length covered by the runners as *iid* realizations from a uniform distribution, find an unbiased estimate of θ based on a minimal sufficient statistic.

ii. Find the BLUE of θ.

iii. Compare the estimators found in (i) and (ii) above with proper justification.

[7+(5+6+2) = 20]

8. (a) The following model is assumed for weather prediction:

(1) If it is *rainy* today then the probability that it would be *rainy*, *cloudy* (but not *rainy*) or *sunny* tomorrow are 0.5, 0.3, 0.2 respectively.

(2) If it is *cloudy* today then the probability that it would be *rainy*, *cloudy* or *sunny* tomorrow are 0.2, 0.4, 0.4 respectively.

(3) If it is *sunny* today then the probability that it would be *rainy*, *cloudy* or *sunny* tomorrow are 0.1, 0.1, 0.8 respectively.

i. Find the transition matrix.

ii. If today is *cloudy*, then find the probability that it will rain the day after tomorrow.

iii. Suppose it is known that tomorrow can either be *rainy* or *cloudy* with equal probability. Can you obtain the distribution of the weather status for the day after tomorrow?

(b) Obtain the regression equation of Y on X for the distribution given by

$$f(x,y) = \frac{ye^{-\frac{y}{(1+x)}}}{(1+x)^4}, \quad x, y \geq 0.$$

[(2+4+4)+10 = 20]

9. (a) Given a single observation from the population with probability density function

$$f(x) = \frac{2}{\theta^2}(\theta - x),$$

find $100(1-\alpha)\%$ confidence interval for θ.

(b) Suppose $X_i,\ i = 1, 2, \cdots, n$ follows independent $N(\beta_i\theta, \beta_i\sigma^2)$, where β_i, σ^2 are known positive quantities.

i. Find the uniformly most powerful test of level α for testing

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta = \theta_1 > \theta_0.$$

ii. Examine whether uniformly most powerful test of level α exists for $H_1 : \theta \neq \theta_0$.

(c) Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{for } 0 \leq x \leq \theta \\ 0 & \text{otherwise.} \end{cases}$$

Consider the following two critical regions based on a single observation x : $C_1 = \{x \geq 0.5\}$ and $C_2 = \{1 \leq x \leq 1.5\}$, for the problem of testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$. Find out the probability of type I error of the two tests and their powers.

[5+(6+3)+6 = 20]

10. (a) Suppose a fair coin is tossed 144 times. Find an approximate value of the probability that the number of heads is at most 66.

[Assume $\Phi(0.5) = 0.84$, $\Phi(1) = 0.69$, $\Phi(1.5) = 0.93$, $\Phi(2) = 0.98$]

(b) Let $X_1, X_2, \ldots$ be *iid* samples from exponential distribution with mean λ. Define $Y_i = {X_i}^2$, $i = 1, 2, \cdots$. Find a_n and b_n such that

$$\frac{\bar{Y}_n - a_n}{b_n} \xrightarrow{d} Z, \quad \text{as} \quad n \to \infty,$$

where $\bar{Y}_n = (Y_1 + \cdots + Y_n)/n$, Z is distributed as N(0,1), and $\xrightarrow{d}$ indicates convergence in distribution.

(c) Let $W_1 < W_2 < \cdots < W_n$ be the order statistics of n independent observations from a U(0,1) distribution.

i. Show that the probability density function of W_r follows Beta distribution.

ii. Show that

$$E({W_r}^2) = \frac{r(r+1)}{(n+1)(n+2)}.$$

[4+6+(5+5) = 20]

Part II: Section E (Engineering Stream)

(Answer any **FOUR** questions.)

5. (a) A car stood in front of a tea shop. A passenger bus traveling with a constant velocity of 54 km/h crossed the car. The car started after 5 seconds and accelerated by 25 m/s^2 until it reached the velocity of 72 km/h. It continued with the same velocity and eventually crossed the bus after some time. What distance does the car travel before it overtakes the passenger bus?

   (b) A thin cylinder of diameter d and thickness t is pressurized with an internal pressure of p. Assume that E is the modulus of elasticity and ν is the Poisson's ratio. If $\nu = 0.2$, find the ratio of longitudinal strain to circumferential strain.

   (c) Find the moment of inertia of the shaded region, as shown in Figure 2, about X and Y axes.

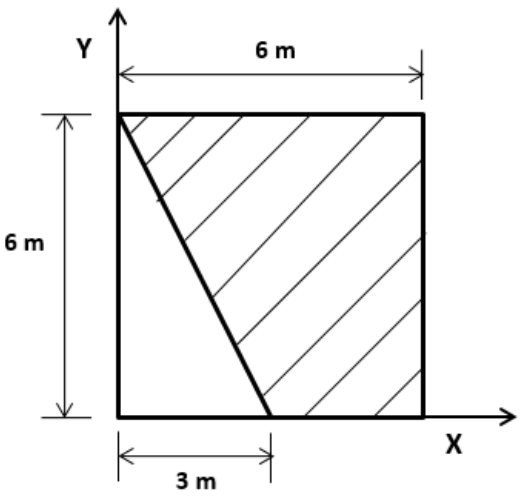

Figure 2

[6+6+8 = 20]

6. (a) An induction motor drives the main shaft of a workshop by means of a flat belt. The diameter of the pulleys on the motor shaft and the main shaft are 0.4 m and 0.8 m respectively. Another pulley of diameter 0.7 m, which is keyed to the main shaft, drives a countershaft having a pulley of diameter 1.4 m keyed to it. If there is a 2% slip on each drive, determine the speed of the countershaft if the motor runs at 1400 rpm. Neglect thickness of the belt.

   (b) A person does 30 kJ work on 2 kg of water by stirring with the help of a paddle wheel. While stirring, around 20 kJ of heat is released from water through its container to the surface and surroundings by thermal conduction and radiation. What is the change in internal energy of the system?

   (c) An inventor claims that his new engine will develop 30 kW for a heat addition of 2400 kJ/min. The highest and lowest temperatures of the cycle are 2200 °C and 327 °C respectively. Would you agree to his claim?

[10+5+5 = 20]

7. (a) A system undergoes a cycle involving three processes, with details given in the table below, with the values of heat (Q), work (W) and change in internal energy (ΔU).

Process	Q(kJ)	W(kJ)	ΔU(kJ)
$1-2$	a	150	150
$2-3$	b	-75	c
$3-1$	150	d	-300

   Find out the values of a, b, c and d.

   (b) Butane is burned in dry air. The volumetric analysis of the products is found as 11% CO_2, 1% CO, 3.5% O_2 and

84.5% N_2. Determine the percent of theoretical air used in combustion.

(c) For heating water, an electric coil with a resistance of 40 Ω is connected to a power source of 240 V for one hour. During the process, the temperature of the heater remains constant at 90 °C. Determine

i. the work done by the heater,

ii. the heat transferred to the heater,

iii. change in entropy of the heater and the water,

iv. entropy change of the universe.

[4+10+(2+1+2+1) = 20]

8. (a) For the circuit given in Figure 3, assume that the diode D is ideal, $R_1 = 2\ k\Omega$, $R_2 = 8\ k\Omega$, $R = 2\ k\Omega$, $R_3 = 8\ k\Omega$ and $v_{s2} = 0.75$ V. Find the output voltage v_0 for *i*) $v_s = 1.5$ V and *ii*) $v_s = -1.5$ V.

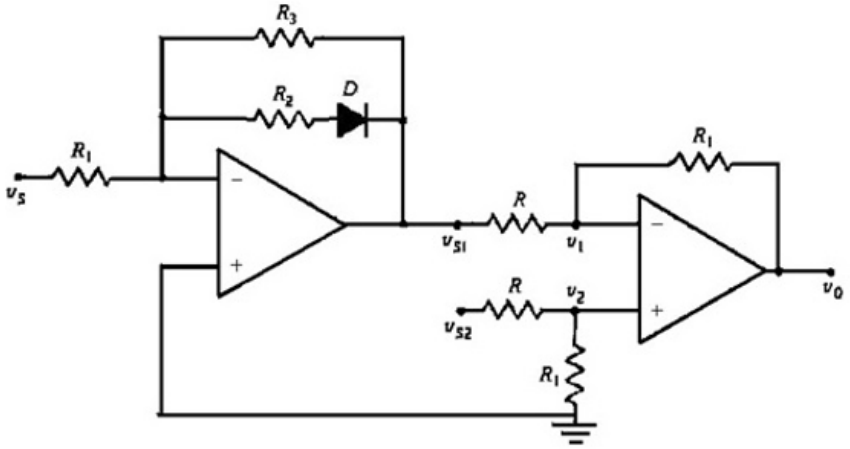

Figure 3

(b) i. Explain the operation of a half adder.

ii. Show that a full adder can be constructed with two half adders and an OR gate using the necessary equations, truth tables and circuit diagrams.

[(6+4)+(4+6) = 20]

9. (a) A 4-pole 404 V wave winded DC shunt motor is running at 1200 rpm. It has 400 copper conductors and draws 12 A from the supply. The armature resistance is 0.4 Ω and the shunt field resistance is 202 Ω.

i. Find the values of armature current, back emf and flux per pole.

ii. Calculate the speed at which the motor will run if 3.6 Ω resistor is introduced in the series with the armature.

(b) For the circuit shown in Figure 4,

i. Obtain the value of R such that maximum power will be transferred to R.

ii. Also prove that, the value of R thus obtained, corresponds to the maximum power.

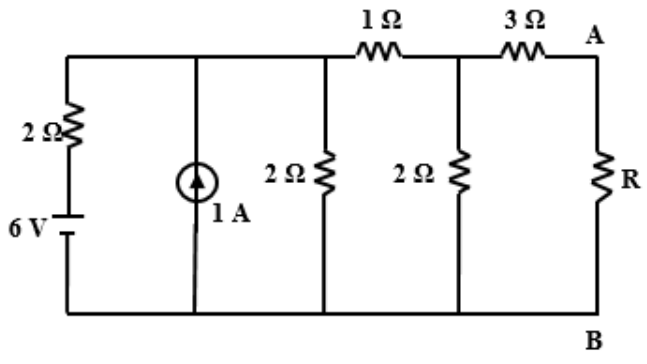

Figure 4

[(8+4)+(4+4) = 20]

10. Draw the top and the front views of a regular hexagonal prism of base side 25 mm and axis length 50 mm when it is lying on the ground on one of its rectangular faces and the axis is inclined at 45° to the vertical plane.

[20]