Part I: Section M (Common for Both Streams)

(Answer any **TWO** questions)

1. (a) Suppose that p is an odd integer greater than 3, but not divisible by 3. Show that $p^2 - 1$ is divisible by 24.

   (b) Sketch the graph of the function $f(x) = \frac{x}{1+x^2}$, $x \in (-\infty, \infty)$.

   (c) Show that for positive integer n,

   $$\left[\binom{n}{0} + \binom{n}{1}\right]\left[\binom{n}{1} + \binom{n}{2}\right] \cdots \left[\binom{n}{n-1} + \binom{n}{n}\right]$$
   $$= \frac{(n+1)^n}{n!}\binom{n}{0}\binom{n}{1}\cdots\binom{n}{n-1}.$$

   $$[8 + 7 + 5 = 20]$$

2. (a) From a point h meters above the surface of a lake, the angle of elevation of a cloud is θ and the angle of depression of its reflection in the lake is ϕ. Find the height of the cloud above the lake surface.

   (b) Consider the pair of straight lines given by

   $$x^2(\tan^2 \theta + \cos^2 \theta) - 2xy \tan \theta + y^2 \sin^2 \theta = 0.$$

   Show that the difference between the gradients of the lines is 2.

   $$[10 + 10 = 20]$$

3. (a) If $f(x) = \frac{\sin 2x + A \sin x + B \cos x}{x^3}$ is continuous at $x = 0$, find the values of A, B and $f(0)$.

   (b) Find the volume of the largest cylinder that can be placed inside a hollow right circular cone with base radius R cm and height H cm.

1

(c) Find the area of the region bounded by the curves $y = |x|$ and $y = 2x + x^2 - 2x^3$.

$$[6 + 8 + 6 = 20]$$

4. (a) Let $p(x) = x^4 - a_1 x^3 + a_2 x^2 - a_3 x + 1470$ be a polynomial such that all roots of the equation $p(x) = 0$ are positive integers greater than unity. Further suppose that

$$|p(7)| + |p^{'}(7)| = 0, \text{ where } p'(x) = \frac{d}{dx} p(x).$$

(i) Find the least and greatest possible values of a_1.

(ii) Find all the possible polynomials.

(iii) From the set of polynomials found in (ii) above, write down the polynomial which has the least value for any $x \in \mathbb{R}$.

(b) Consider the following matrix for $a, b, c \in \mathbb{R}$,

$$A = \begin{pmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{pmatrix}.$$

(i) Find $det(A)$.

(ii) If $a^2 + b^2 + c^2 = 4$, find the maximum and minimum numerical values of the above determinant.

$$[(4 + 3 + 5) + (3 + 5) = 20]$$

2

5. (a) Let $A = \{1, 2, \ldots, 49, 50\}$. Suppose two numbers are chosen randomly with replacement from the set A. Find the probability that their difference is divisible by 3.

   (b) A slip of paper is given to A, who marks it with either a plus sign $(+)$ or a minus sign $(-)$. The probability of marking plus sign is $\frac{1}{3}$. Thereafter, the slip is passed on to B, C and D sequentially. Each of them either alters the sign with probability $\frac{2}{3}$ or leaves it unchanged. Finally D hands over the slip to the referee, who finds plus sign on it. What is the probability that A has marked plus sign on the slip?

$$[10 + 10 = 20]$$

6. (a) The joint probability density function of (X, Y) is given by

$$f(x, y) = \frac{1 + xy}{4}, \quad -1 < x, y < 1.$$

   Find the joint probability density function of (X^2, Y^2).

   (b) Let $X_0, X_1, \ldots$ be a Markov chain with state-space $\{1, 2, 3\}$ and transition probability matrix

$$P = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} & 0 \\ \frac{2}{5} & \frac{1}{2} & \frac{1}{10} \\ 0 & \frac{1}{10} & \frac{9}{10} \end{bmatrix}.$$

   (i) Find the probability distribution of X_1, if the probability distribution of X_0 is $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$.

   (ii) Classify the states.

$$[10 + (6 + 4) = 20]$$

7. (a) Suppose the number (N) of customers that arrive at an ATM to withdraw money, during a given period of one hour, has a Poisson distribution with mean 4. The amount (X) of money withdrawn by any customer is a discrete random variable with probability mass function

$$p_X(x) = \frac{1}{5}, \quad \text{if } x = 2000, 4000, 6000, 8000, 10000.$$

Assume that N and X are independent.

   (i) Find the expected total amount of money drawn during the period given that at least one customer withdrew money.

   (ii) What is the probability that 3 customers have withdrawn money given that the total withdrawal is Rs. 8000 during the period?

   (b) Let X be a random variable with probability mass function

$$P(X = x) = \begin{cases} e^{-1} & \text{if } x = 0 \\ \dfrac{e^{-1}}{2(|X|)!} & \text{if } x \in \{\pm 1, \pm 2, \ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

   Find the distribution of $Y = |X|$.

$$[(6 + 6) + 8 = 20]$$

8. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Find the UMVUE of μ^3, if σ^2 is known.

   (b) A salesman of used cars assumes that the number of sales he makes per day is a Poisson random variable with parameter μ. Over the past 30 days, he made no sales on 20 days and one or more sales on each of the remaining 10 days. Find the maximum likelihood estimate of μ.

$$[12 + 8 = 20]$$

4

9. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics corresponding to a random sample from the distribution with probability density function

$$f(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty, \quad \lambda > 0.$$

Consider the transformations

$$Z_1 = nX_{(1)}, Z_i = (n - i + 1)(X_{(i)} - X_{(i-1)}), i = 2, 3, \ldots, n.$$

(a) Show that $Z_1, Z_2, \ldots, Z_n$ are iid with pdf f.

(b) Hence or otherwise find mean and variance of sample range.

$$[10 + (4 + 6) = 20]$$

10. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = \theta(1 - x)^{\theta - 1}, \quad 0 < x < \infty, \quad \theta > 0.$$

(i) Derive an α-level most powerful test for $H_0 : \theta = 1$ versus $H_1 : \theta > 1$. Find the power function of the test.

(ii) Derive a $100(1 - \alpha)\%$ confidence interval for θ.

(b) Consider the following linear model under the Gauss-Markov set-up:

$$E(Y_1) = \beta_2$$
$$E(Y_2) = 2\beta_1$$
$$E(Y_3) = \beta_1 + \beta_2.$$

Derive least-square estimators of (β_1, β_2). Verify whether they are unbiased.

$$[(8 + 3) + (6 + 3) = 20]$$

Part II: Section E (Engineering Stream)

(Answer any **FOUR** questions)

5. (a) Three solid spheres, each of mass m and diameter d, are glued together in a manner that the lines connecting the centers form an equilateral triangle of side of length d as shown in Figure 1. The centroid of the system is located at point O. Now, I_O and I_A are the moments of inertia of the system about the axes (perpendicular to the plane of the aforementioned triangle) passing through the centroid O and point A, respectively. Find out the value for the ratio: I_O/I_A.
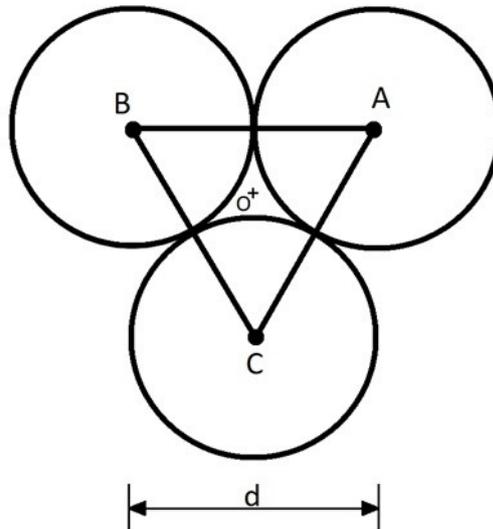
Figure 1

(b) A particle of mass m is projected vertically upwards with a velocity v_0 in a medium whose resistance is kv where v is the instantaneous velocity and k is a constant. Determine the expression of time for the particle to come to rest.

6

(c) Consider the stress elements as shown in Figure 2. The values of σ_x, σ_y and τ_{xy} are -10 MPa, 20 MPa and -20 MPa respectively. Find out the values of the principal stresses.
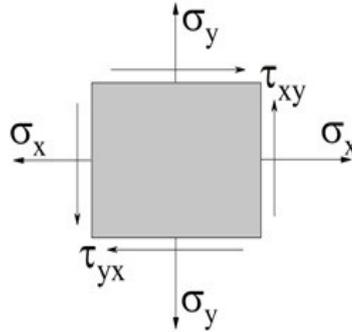

Figure 2

$$[8 + 6 + 6 = 20]$$

6. (a) A heat pump operates between two identical bodies which are initially at temperature T_1 and cools one of the bodies to a temperature T_2 $(T_2 < T_1)$. Prove that for this operation the minimum work required by the heat pump is given by

$$W = C_p \left(\frac{T_1^2}{T_2} + T_2 - 2T_1 \right)$$

where C_p is the specific heat for both the bodies.

(b) Two flat blocks are resting on the inclined plane and a force P is applied to the 10 kg lower block parallel to the inclined plane in the direction shown in Figure 3. The 5 kg upper block is prevented from moving by a cable, attached to a fixed support. The corresponding coefficients of static friction for the two mating surfaces are shown in Figure 3. Considering $g = 10$ m/s², determine

(i) the tension in the cable,

7

(ii) the force that makes the 10 kg lower block start sliding downward,

(iii) the feasibility of moving the blocks together upward by applying a commensurate force parallel to the inclined plane.
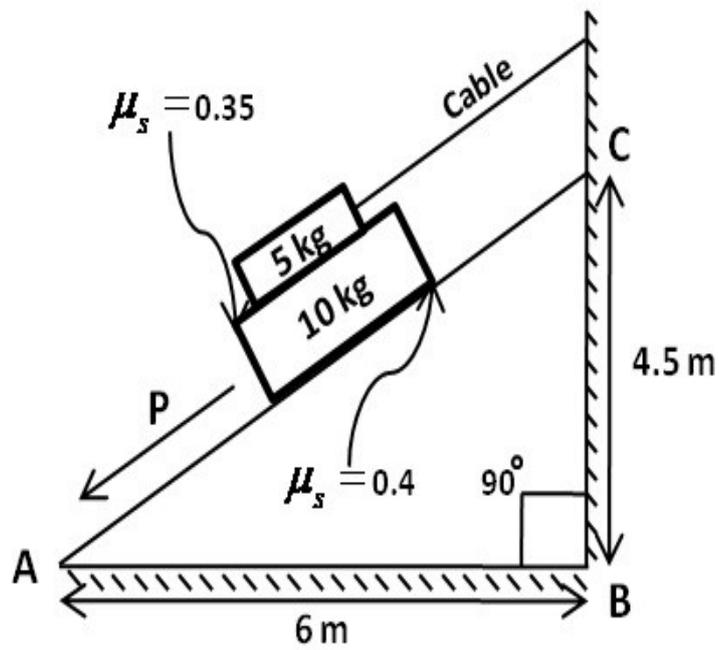

Figure 3

$$[10 + (4 + 4 + 2) = 20]$$

7. (a) Determine the entropy change of 4 kg of a perfect gas whose temperature varies from 127 °C to 227 °C during a constant volume process. The specific heat varies linearly with absolute temperature and is represented by the relation:

$$C_v = (0.48 + 0.0096T) \text{ kJ/kgK}.$$

$[\ln 1.25 = 0.2 \ (approx.)]$

8

(b) Source 1 can supply energy at the rate of 12,000 kJ/min at 343 °C. A second source 2 can supply energy at the rate of 1,20,000 kJ/min at 69.3 °C. Which source (1 or 2) would you choose to supply energy to an ideal reversible heat engine to produce large amount of power if the temperature of the surroundings is 35 °C?

(c) 281.5 kW of heat is supplied at a temperature of 290 °C to a heat engine. The heat rejection takes place at 7 °C. The following results are obtained:

 (i) 280 kW is rejected,

 (ii) 140 kW is rejected,

 (iii) 70 kW is rejected.

Classify the results as reversible cycle or irreversible cycle or impossible cycle.

$$[5 + 6 + (3 + 3 + 3) = 20]$$

8. (a) For the circuit shown in Figure 4, find the voltage across and current through the load resistor 3 Ω using Norton's theorem.
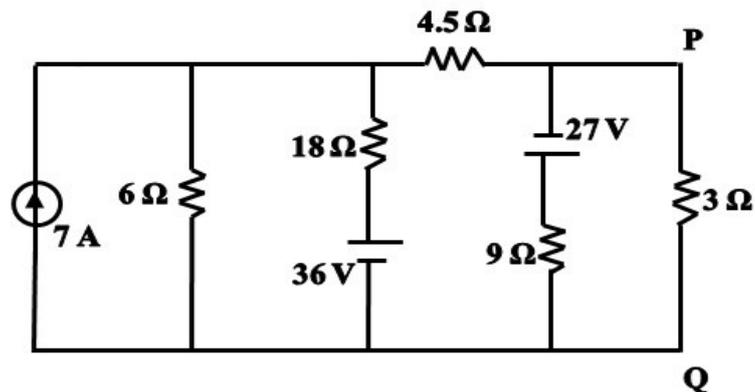

Figure 4

(b) A rotor of 440 V, 50 Hz, 4-pole, 3-phase induction motor is observed to make 120 complete alterations per minute.

9

(i) What will be the rotor speed?

(ii) If the power input to the rotor is 90 kW, what will be the rotor copper loss per phase?

(c) In a 300 kVA, $1000\sqrt{3}$ V, 50 Hz, 3-phase star connected alternator, the effective armature resistance is 3 Ω per phase. A field current of 40 A produces a short circuit current of 100 A and an open circuit emf of $500\sqrt{3}$ V. Find the percentage voltage regulation using emf method at full load with 0.6 lagging power factor.

$$[8 + (4 + 2) + 6 = 20]$$

9. (a) For the circuit given in Figure 5, find

(i) the resistance between the terminals A and B using star delta conversion,
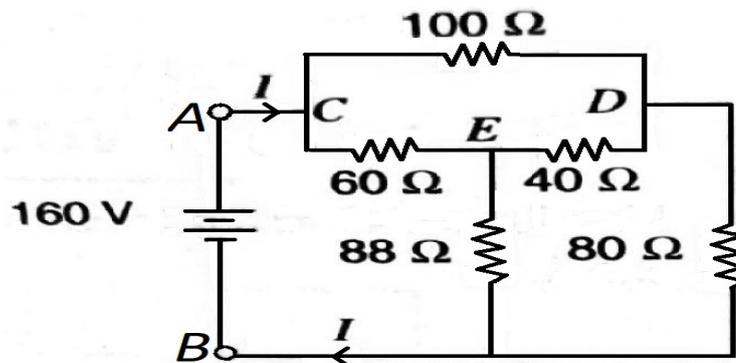
(ii) the current I.

Figure 5

(b) (i) Draw the circuit diagram of a 4-bit shift register using D-type flip-flops.

(ii) Explain how to store a binary serial data 1011 in the shift register mentioned in (i).

$$[(7 + 2) + (4 + 7) = 20]$$

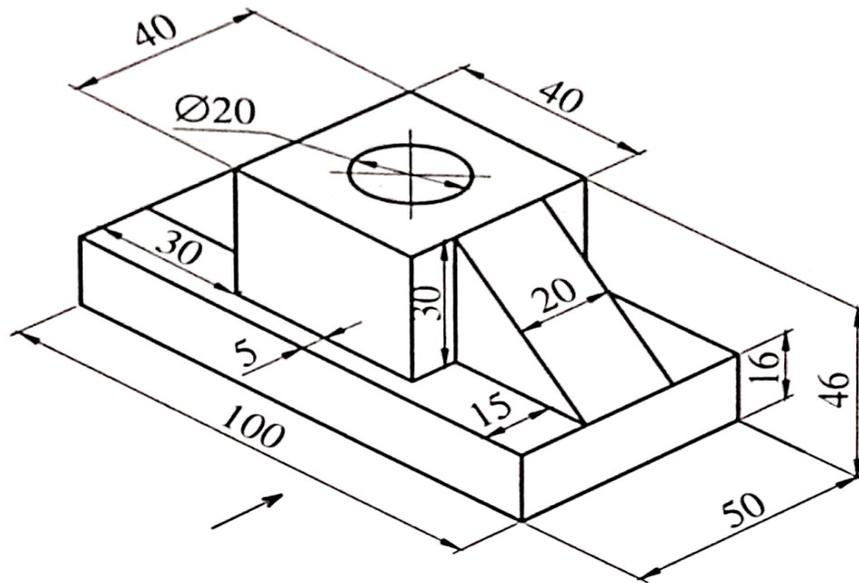10. Draw the top view and front view of the object shown in Figure 6 using third angle projection.

Figure 6

$[10 + 10 = 20]$