PART I : Section M (Both Streams)[1]

(Answer any TWO questions)

1. (a) A positive integer is called palindrome if it reads the same forward or backward. Find the number of 8-digit palindromes divisible by 5.

   (b) Consider the set $S = \{1,2,\cdots,100\}$ of numbers. Find the number of non-empty subsets T of S such that the product of all the numbers in each such subset T is even.

   (c) If n is a positive integer, then prove that the integral part of $\left(3+\sqrt{7}\right)^n$ is an odd number.

   [5 + 5 + 10 = 20]

2. (a) Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$. Find a matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix.

   (b) Show that the series $\cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + \cdots\infty$ converges to $\frac{\pi}{4}$.

   (c) Let A and B be the two nonempty sets for which $n(A) = 7$ and $n(B) = 10$. Find the number of ways by which one-one function can be formed from A to B.

   [8 + 8 + 4 = 20]

3. (a) Find the altitude of a right circular cone, with semi-vertical angle α, of least volume that can be circumscribed around a sphere of radius R. Consequently, find the value of $\sin\alpha$.

   (b) Find the solution to the differential equation $\left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx} = \left(\frac{x+y+1}{x+y+2}\right)$ given that $y = 1$, when $x = 1$.

   (c) Evaluate $\lim_{x\to 0}\frac{\ln(1+2x)-2\ln(1+x)}{x^2}$ without using L'Hospital's Rule.

   [(8+2) + 6 + 4 = 20]

---

[1] Mathematics is now compulsory for Statistics Stream also.

4. (a) A triangle ΔABC, right-angled at C, moves such that the vertices A and B always lie in the first quadrant of the X-Y plane. Prove that the locus of C is a straight line.

(b) A variable line drawn through the point of intersection $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ meets the axes at A and B. Show that the locus of the midpoint of AB is the curve $2xy(a+b) = ab(x+y)$.

(c) Suppose that $\{a_n\}$ is a sequence of numbers such that $a_n > 0$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{6}$. Does the series $\sum \frac{a_n}{1+a_n}$ converge? Justify.

[8 + 7 + 5 = 20]

PART II : Section S (Statistics Stream)

(Answer any FOUR questions)

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution over $(0, \beta)$.

(a) Find an unbiased estimator of β based on largest order statistic $X_{(n)} = \max(X_1, \ldots, X_n)$.

(b) Find an unbiased estimator of β based on sample mean $\bar{X}$.

(c) Compare the relative efficiency of the unbiased estimators obtained in (a) and (b).

[7 + 3 + 10 = 20]

2. (a) Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution with mean λ and $Y_1, \ldots, Y_n$ be a random sample from another exponential distribution with mean μ. Assume that X_i's and Y_i's are independent.

(i) Derive the sampling distribution of $\bar{X}/\bar{Y}$, where $\bar{X}$ and $\bar{Y}$ are the sample means based on X_i's and Y_i's.

(ii) Find a $100(1-\alpha)\%$ confidence interval for $\frac{\lambda}{\mu}$.

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, 1)$.

(i) Show that the estimator $\bar{X}^2 - \frac{1}{n}$ is unbiased and consistent estimator of μ^2, where $\bar{X}$ is the sample mean.

(ii) Find the Cramer-Rao lower bound of an unbiased estimator of μ^2.

[(8+2) + (5+5) = 20]

3. A study was conducted to assess whether job satisfaction is independent of gender. Accordingly, 150 females and 400 males were interviewed and the observations are given below

Gender	Dissatisfied	Neutral	Fairly Satisfied	Completely Satisfied
Female	20	40	40	50
Male	60	40	100	200

(a) Estimate the proportion of males whose job satisfaction is higher than that of females and the proportion of females whose job satisfaction is higher than males.

(b) Assume that the proportions of males whose job satisfaction is higher than that of females and the proportion of females whose job satisfaction is higher than males may be construed as the probabilities of these events.

(i) Write down the probability statements for the two events given above.

(ii) Explain how you will use these probabilities to judge whether gender and job satisfaction are independent.

[10 + (5+5) = 20]

4. (a) There are 25 similar balls in a box, numbered 01 to 25. Five balls would be drawn randomly from the box. You have to guess the number(s) on these 5 balls, before the draw starts.

(i) What is the probability that you get all of them right?

(ii) What is the probability that you get at least 3 of them right?

(b) Suppose $X \sim N(\mu, \sigma^2)$. Show that $E[\Phi(a + bX)] = \Phi\left(\frac{a+b\mu}{\sqrt{1+b^2\sigma^2}}\right)$, where Φ is the cumulative distribution function of standard normal variable and, a and $b(\neq 0)$ are constants.

[(5+5) + 10 = 20]

5. (a) Suppose $X \sim Poisson(1)$ and $Y \sim Binomial\left(2, \frac{1}{2}\right)$ are two independent random variables. Find $P(X \geq Y)$.

(b) Suppose $X = (X_1, X_2, X_3)' \sim N_3(\mu, \Sigma)$, where $\mu = (5, 4, 3)'$ and

$$\Sigma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(i) Find the conditional distribution of $(X_2 - X_1, X_3 - X_2)$ given $X_1 = x_1$.

(ii) Are $X_2 - X_1$ and $X_3 - X_2$ independently distributed? Justify your answer.

[5 + (12+3) = 20]

6. (a) A certain drug is tested chemically for its toxic nature. Let the event "*the drug is toxic*" be denoted by E and the event "*the test reveals that the drug is toxic*" be denoted by F.

Let $P(E) = \theta$, $P(F|E) = P(F^c|E^c) = 1 - \theta$. Show that the probability that the drug is not toxic given that the test reveals that it is toxic is free from θ.

(b) An unbiased coin is tossed $(m + n)$ times $(m > n)$. Find the probability of occurrence of at least m consecutive heads.

[10 + 10 = 20]

PART II : Section E (Engineering Stream)

(Answer any FOUR questions)

1. (a) Two equal parallel forces, acting in opposite direction at fixed points A and B, form a couple of moment G. If their lines of actions are turned through a right angle, they form a couple of moment H. Show that when they both act at right angles to AB, they form a couple of moment $\sqrt{G^2 + H^2}$.

   (b) There is a solid right circular cylinder whose radius is a and altitude is h. From this cylinder, a right cone of the same radius and altitude is cut. If M is the mass of the remaining solid, show that the moment of inertia of the remaining solid about the axis of the cylinder is $\frac{3}{5}Ma^2$.

   (c) A heavy chain of length 100 m and weighing 10 kgm^{-1} hangs vertically with one end attached to a peg and carries a weight of one tonne at the other end. Find the work done in winding 50 m of the chain round the peg.

   [4 + 8 + 8 = 20]

2. (a) It is well known that when NaCl is added to water then the temperature of the solution decreases but if we add $CaCl_2$ instead of NaCl, then the temperature of the solution increases. Explain the reasons behind this contrasting behavior, indicating the changes involved in ΔH, ΔS and ΔG in each of the two cases (notations have their usual meaning).

   (b) The polymerization of ethylene is a spontaneous process. Explain whether the sign of the enthalpy change will be positive or negative during polymerization.

   (c) Standard vaporization enthalpy of water is 40.657 kJmol^{-1} at 100°C. How long would it take to vaporize a sample of 100 g water at the boiling point using a 100 W electric heater?

   (d) A 1.25 g sample of octane is burned in excess oxygen in a bomb calorimeter. The temperature of the calorimeter rises from 21°C to 31°C. If the heat capacity of calorimeter is 8.93 kJK^{-1}, calculate the enthalpy of combustion of octane. Given that combustion equation is $2C_8H_{18}(l) + 25O_2(g) = 16CO_2(g) + 18H_2O(l)$.

(e) Explain the difficulty of eliminating environmental pollution using the argument of entropy changes.

[3 + 2 + 5 + 8 + 2 = 20]

3. (a) From the first law of thermodynamics, show that the change in entropy (ΔS) of a perfect gas having mass m is

$$\Delta S = m\left[C_V \ln\left(\frac{T_2}{T_1}\right) + (C_P - C_V)\ln\left(\frac{V_2}{V_1}\right)\right]$$

where

$V_1 =$ initial volume of the gas

$T_1 =$ initial temperature of the gas

$V_2, T_2 =$ corresponding values for the final condition

$C_P =$ specific heat at constant pressure

$C_V =$ specific heat at constant volume.

(b) An engineer claims to have designed a device which absorbs 2500 kJ of heat and produces 2000 kJ of work. If the heat sink for the device is maintained at 27^0C, what would be the minimum source temperature?

(c) In a workshop, an engine drives a shaft by a belt. The diameters of the engine pulley and the shaft pulley are 500 mm and 250 mm respectively. Another pulley of 700 mm diameter on the same shaft drives a pulley of 280 mm diameter on the motor shaft. If the engine runs at 180 rpm, find the speed of the motor shaft.

[8 + 7 + 5 = 20]

4. (a) Show that the maximum power obtainable from a source having an open circuit voltage E and an internal impedance $(R + jX)\Omega$ is given by $\left(\frac{E^2}{4R}\right)$ W.

(b) Calculate the average and rms values of current represented in Figure 1 below.

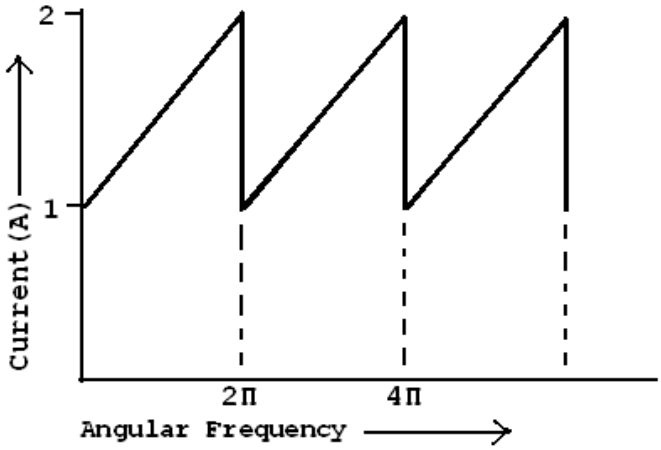

Figure 1

[12 + 8 = 20]

5. (a) In the circuit, represented in Figure 2 below, express v_0 in terms of v_{S1}, v_{S2} and other depicted parameters.

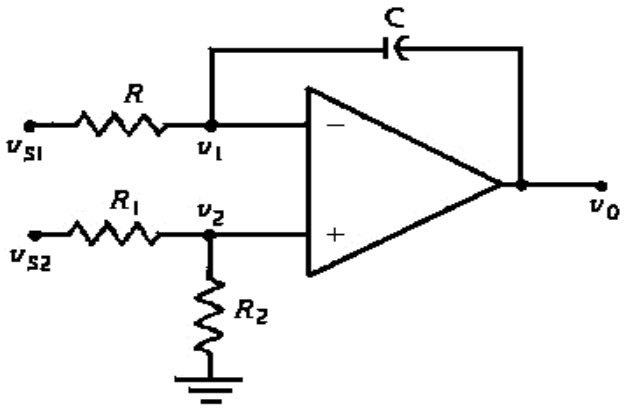

Figure 2

(b) Subtract 10101 from 10011 using 2's complement method.

(c) (i) Apply Thevenin's theorem to derive an expression for the current flowing through $X\Omega$ in Figure 3 below.

(ii) Determine the value of X in Figure 3 for which the maximum power is dissipated in the resistance.

(iii) Determine the value of the dissipated power.

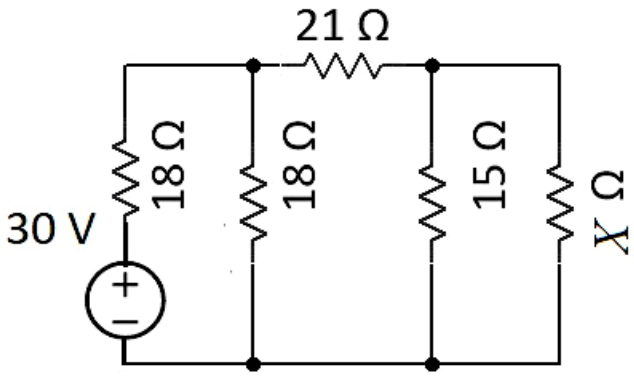

Figure 3

[5 + 3 + (6+4+2) = 20]

6. (a) A cylinder with a base of 40 mm diameter and 50 mm long rests on a point of its base on HP such that the axis makes an angle of 30° with HP. Draw the front view and top view of the cylinder. Follow the rules of the orthographic projection.

(b) Draw the isometric view from the front view and top view of an object provided below in Figure 4.

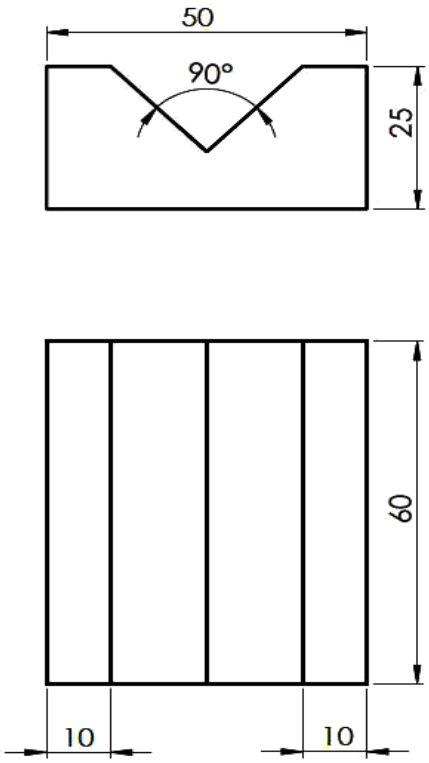

Figure 4

[12 + 8 = 20]