

PART I (STATISTICS / MATHEMATICS STREAM)

ATTENTION: Answer a total of **SIX** questions taking **at least TWO** from each Group - **S1 & S2**.

GROUP S1: Statistics

1. (a) Let X_1, X_2, \dots, X_n be a random sample from $N(\theta_1 + \theta_2, 1)$, and Y_1, Y_2, \dots, Y_n be a random sample from $N(\theta_1 - \theta_2, 1)$ independent of the first one. Find the maximum likelihood estimators of θ_1 and θ_2 . Derive a $100(1 - \alpha)\%$ confidence interval for θ_1 .

- (b) Let X_1, X_2, \dots, X_n be a random sample from the distribution with probability density function

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for $\lambda > 0$. Find a uniformly most powerful level α test for testing $H_0: \lambda = \lambda_0$ versus $H_1: \lambda > \lambda_0$. Provide the power function of the test.

[(5+5) + 10 = 20]

2. A random sample of size n is generated from a random variable X having probability density function

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x > 0 \\ 0, & \text{else} \end{cases}$$

for unknown parameter $\theta > 0$. However, for some unknown reason, only observations having value less than A (a known positive constant) are recorded. Let the recorded observations be denoted by x_1, x_2, \dots, x_m for $0 \leq m < n$.

- (a) Write down the likelihood function. Justify.
(b) Find a sufficient statistic for θ .
(c) Provide the condition under which maximum likelihood estimator (*mle*) of θ will exist.
(d) Obtain *mle* of θ under the condition stated above in (c).

[6 + 6 + 3 + 5 = 20]

3. (a) Let X_1, X_2, \dots, X_n be a random sample from $N(0, \sigma^2)$. Find an unbiased estimator of σ based on the statistic $T = \frac{1}{n} \sum_{i=1}^n |X_i|$. Find its Cramer-Rao lower bound.

(b) Suppose that X_1, X_2, \dots, X_n constitutes a random sample from a distribution having mean μ and variance σ^2 . Let $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$ be sample mean and variance respectively. Show that $E(S) \leq \sigma$.

[(7+7) + 6 = 20]

4. (a) Consider a random sample X_1, X_2, \dots, X_n from the discrete distribution:

$$P(X = 1) = \frac{2(1-\theta)}{2-\theta}, \quad P(X = 2) = \frac{\theta}{2-\theta}$$

for unknown $\theta \in (0,1)$.

(i) Find a moment estimator of θ .

(ii) Is this estimator consistent? Justify your answer.

(b) Suppose that in a population of size $N = 800$, P (unknown) is the percentage of people having a particular blood group. In order to estimate P , a random sample of size n is to be drawn with replacement.

(i) Determine the value of n so that P can be estimated within an absolute error of 4%, with probability 0.95.

(ii) If you have prior knowledge that P is less than 30%, would you consider changing the value of n obtained in (i)? Justify your answer.

[(5+5) + (6+4) = 20]

5. (a) Consider the linear model: $y_{ij} = \mu + \alpha_i + \epsilon_{ij}$ for $i = 1, 2, 3$, $j = 1, 2, 3$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$.
- (i) Derive least-square estimators of the unknown parameters μ, α_1, α_2 and α_3 . Are these estimators unique? Justify your answer.
- (ii) If the condition $\alpha_1 + \alpha_2 + \alpha_3 = 0$ is dropped, what can you say regarding uniqueness of the above estimators?
- (b) Suppose that the average of 100 numbers (not all equal) is zero. Some of these numbers are 0's. Remove all these 0's from the original set. Explain with justification, whether variance of the remaining numbers is *larger than*, *less than* or *equal to* the variance of the original set of numbers.

$$[(5+5) + 10 = 20]$$

GROUP S2: Probability

6. (a) Consider a square having each side as 5 cm. Pick a random point inside it. Denote the four perpendicular distances of the sides, from this point, by X_1, X_2, X_3 and X_4 .
- (i) Give a graphical representation of the above, and write down the relationship among the four distances, if there is any.
- (ii) Let Y be the distance of the furthest side from the point picked. Write Y as a function of X_1, X_2, X_3 and X_4 .
- (iii) Hence or otherwise, determine the probability that Y is at least p cm.
- (b) There are three chests, each having two drawers. Each drawer of Chest-1 contains a gold coin, and each of Chest-2 has a silver coin. In the Chest-3, there is a gold coin in one drawer and a silver coin in the other. A chest is selected at random. Then one of its drawers is selected randomly again, and is opened. It is found to contain a gold coin. What is the probability that the other drawer of the selected chest also contains a gold coin?

$$[(3+2+5) + 10 = 20]$$

7. (a) A parent particle gets split into 0, 1 or 2 child particles with probability $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$ respectively, and then disappears immediately. Beginning with 1 particle as the progenitor, let us denote X_i = number of particles in the i^{th} generation, find

(i) $P[X_2 > 0]$,

(ii) $P[X_1=2 \mid X_2=1]$.

- (b) Two athletic teams A and B play a series of independent games between them until one of them wins 4 games. The probability of every team winning any single game is $\frac{1}{2}$. Find the probability that the series will end in at most 6 games.

[10 + 10 = 20]

8. (a) Suppose that X_1, X_2, \dots, X_n are i.i.d. uniform distribution over $\{1, 2, \dots, N\}$, where N is a positive integer. Let $M_n = \max \{X_1, X_2, \dots, X_n\}$. Find the probability distribution of M_n .

- (b) Suppose the joint distribution of X_1, X_2, X_3 is $N_3(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & -1 & 3 \end{pmatrix}.$$

- (i) Find the conditional distribution of (X_1, X_3) given X_2 .

- (ii) Find the partial correlation coefficient between X_1 and X_3 for fixed X_2 .

[10 + (7+3) = 20]

9. Suppose that (X, Y) is uniformly distributed over $\{(x, y) \in R^2: x^2 + y^2 < 1\}$. The probability density function of (X, Y) is given by

$$f(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Find the conditional distribution of X given $Y = y$, where $|y| < 1$.

- (b) Find $E[X|Y]$.

- (c) Find the correlation coefficient between X and Y .

- (d) Are X and Y independent? Give reasons.

[8 + 3 + 4 + 5 = 20]

10. (a) The failures of a certain machine occur according to a Poisson process with rate 2 per week. Find the probability that the machine will have at least one failure in each of the first three weeks.
- (b) Consider the set of all 11-letter words that can be formed by arranging the letters in **DISTURBANCE**. Any two such words are called equivalent if they maintain the same relative order of the vowels and no two vowels appear adjacent to each other. For example, **TURDISBANCE** and **TRUSIDBANCE** are equivalent. Find the probability that a randomly picked word from the set will be equivalent to **DISTURBANCE**.

[10 + 10 = 20]

PART II (ENGINEERING STREAM)

ATTENTION: Answer a total of **SIX** questions taking **at least TWO** from each Group - **E1 & E2**.

GROUP E1: Mathematics

1. (a) Find

$$\lim_{x \rightarrow 1} \frac{\sum_{k=1}^{199} x^k - 199}{x - 1}.$$

(b) Let

$$f(x) = \begin{cases} ax - b, & x \leq 1 \\ 3x, & 1 < x < 2 \\ bx^2 - a, & x \geq 2 \end{cases}$$

be continuous at $x = 1, 2$. Find the values of a and b .

(c) Let $p(x)$ be a polynomial of degree 100, with all real coefficients, satisfying $p(x) > 0$ for every real x . Prove that $p(x)$ is a product of quadratic polynomials having real coefficients.

[5 + 5 + 10 = 20]

2. (a) Give the solution set (in compact form) of the following inequality

$$\frac{|x + 3| + x}{x + 2} > 1.$$

(b) Show that

$$\binom{n}{1} - 2\binom{n}{2} + \dots + (-1)^{n-1} n \binom{n}{n} = 0,$$

where $\binom{n}{r}$ is the number of different groups of r objects that can be chosen from a set of n objects.

(c) Show that the series $\frac{x}{1+x^2} + \frac{1}{3}\left(\frac{x}{1+x^2}\right)^3 + \frac{1}{5}\left(\frac{x}{1+x^2}\right)^5 + \dots$ converges for all $x \in \mathbb{R}$.

[8 + 6 + 6 = 20]

3. (a) Suppose $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + x^2 + 1 = 0$. Find the value of

$$\alpha\beta(\alpha + \beta) + \alpha\gamma(\alpha + \gamma) + \alpha\delta(\alpha + \delta) + \beta\gamma(\beta + \gamma) + \beta\delta(\beta + \delta) + \gamma\delta(\gamma + \delta).$$

- (b) Consider the parabola $y^2 = 4ax$. Inscribe an equilateral triangle to it such that the parabola's vertex coincides with one of the vertices of the triangle. Then a circle is inscribed to the above triangle. Work out areas of the

- (i) triangle, and
(ii) circle.

[10 + (5+5) = 20]

4. (a) If a, b, c are all positive, and are the p th, q th and r th terms respectively of a GP, then show that

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$$

- (b) Consider the function

$$f(x) = |x| - \frac{1}{|x|}, \quad x \neq 0.$$

- (i) Draw the graph of the function $f(x)$.
(ii) Now consider the equation

$$|x| - \frac{1}{|x|} = ax + b.$$

Show that there exist real numbers a and b such that this equation has three distinct real roots, each with multiplicity one, and one of them being unity.

[8 + (6+6) = 20]

GROUP E2: Engineering & Technology

Engineering Mechanics and Thermodynamics

5. (a) A uniform wheel of 600 mm diameter, weighing 5 kN rests against a rigid rectangular block of 150 mm height as shown in Figure 1. Considering the entire surface to be smooth, find
- (i) The least pull required through the centre of the wheel just to turn the wheel over the corner A of the block.
 - (ii) The reaction of the block.

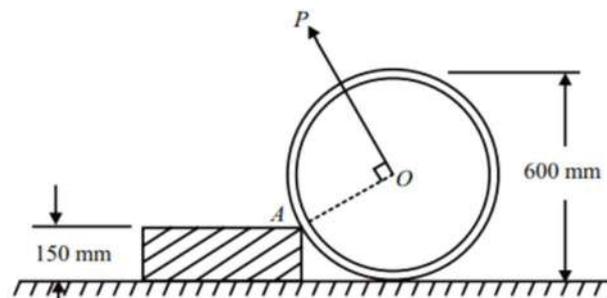


Figure 1

- (b) An electric light fixture weighing 15 N hangs from a point by two strings. One string makes an angle of 45° to the vertical plane and the other string makes an angle of 60° to the horizontal plane. Determine the forces acting on the two strings.
 - (c) An aeroplane requires a velocity of 80 km/hour and a run of 100 m prior to take off. The mass of the aeroplane is 10,000 kg and the frictional coefficient between the wheel and the ground surface is 0.2. Assuming that the aeroplane is accelerating uniformly, find the minimum force required to take off.
- [(7+3) + 5 + 5 = 20]
6. (a) A piston cylinder device is initially at 150 kPa pressure and 27°C temperature. The piston rests on a stop and the volume of air enclosed is 0.4 m^3 . To move the piston, a pressure of 350 kPa is required. The system is heated until the volume is doubled. Determine
- (i) The final temperature of the air.
 - (ii) Work done.

(iii) Total heat transfer.

- (b) A fuel has the composition by mass like: C 90%, H 3%, O 2.5%, N 1%, S 0.5%, and ash 3%. Calculate the volumetric air supply rate required to burn the fuel at the rate of 500 kg/hour considering 20% excess air.

$$[(2+3+5) + 10 = 20]$$

7. (a) Determine if a tray of ice cubes could remain frozen when placed in a food freezer having a COP of 9, operating in a room at a temperature of 32°C .
- (b) It is proposed that solar energy be used to warm a large collector plate. This energy would, in turn, be transferred as heat to a fluid within a heat engine, and the engine would reject energy as heat to the atmosphere. Experiments indicate that $2000\text{ kJm}^{-2}/\text{hour}$ of energy can be collected when the plate is operating at 90°C . Estimate the minimum collector area that would be required for a plant producing 1 kW of useful shaft power. The atmospheric temperature may be assumed to be 25°C .
- (c) Determine the moment of inertia of the object in Figure 2 about both centroidal axes.

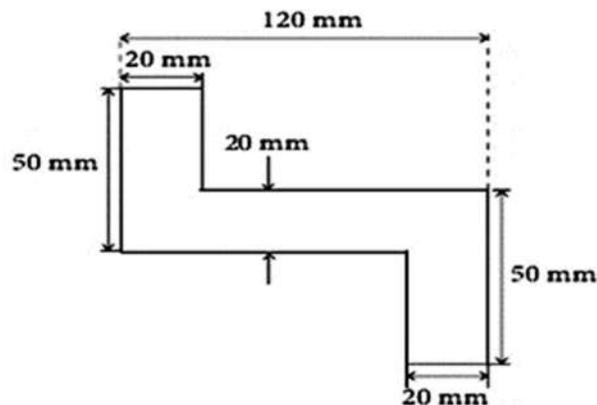


Figure 2

$$[5 + 5 + 10 = 20]$$

Electrical and Electronics Engineering

8. (a) For the given circuit in Figure 3,
- Determine the time constant.
 - Derive the expression of the current " i_L " as a function of time after the switch closes at $t = 0$ second.

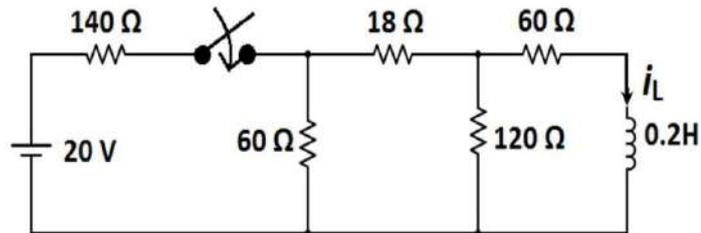


Figure 3

- (b) A 50 Hz single phase transformer has 80 turns on the Low Voltage (LV) winding side and 400 turns on the High Voltage (HV) winding side. The net cross sectional area of the core is 0.02 m^2 . If the LV side is connected to 222 V, 50 Hz supply, determine
- The emf induced on the HV side.
 - The maximum flux density in the core when the HV side is connected to an R-L load, which draws 555 W power at 0.5 lagging power factor.
 - The magnitude of the RMS current flowing through the HV winding and the LV winding, neglecting the no-load current and voltage drop in the winding.

$$[(6+8) + (1+2+3) = 20]$$

9. (a) Simplify the Boolean function $Y = \bar{A}BC + A\bar{B}C + ABC\bar{C} + ABC$ using a Karnaugh map and draw the simplified logic circuit.
- (b) In the npn transistor, shown in Figure 4, 10^9 holes/ μs move from the base to the emitter region while 10^{10} electrons/ μs move from the emitter to the base region. An ammeter reads the base current as $i_B = 16\mu\text{A}$. Considering the charge of 1 electron = $-1.602 \times 10^{-19} \text{ C}$, determine
- The emitter current i_E .
 - The collector current i_C .

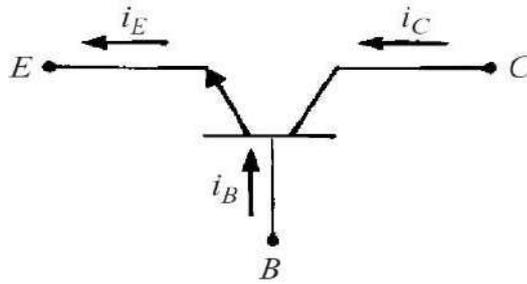


Figure 4

- (c) Subtract 6702 from 8415 using 9's complement method.
- (d) For the circuit given in Figure 5, $V_1 = 12V$, $V_2 = 6V$, $R_1 = 4$ Ohms, and $R_2 = 6$ Ohms. Assuming that the internal resistances of the batteries are zero, find the Thevenin equivalent for the network to the left of the terminals a, b.

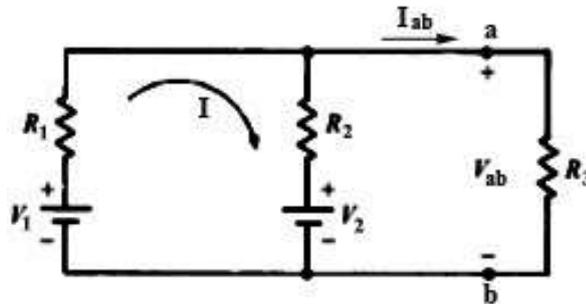


Figure 5

$$[4 + (3.5+1.5) + 3 + 8 = 20]$$

Engineering Drawing

10. (a) Draw the front view and top view of the object shown in Figure 6 using the third angle of projection.

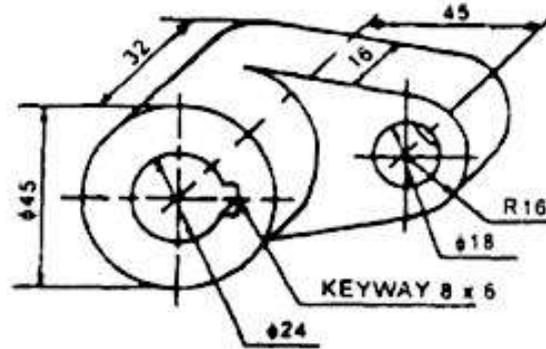


Figure 6

- (b) Draw the isometric view from the projection drawings (in the first angle) as shown in Figure 7.

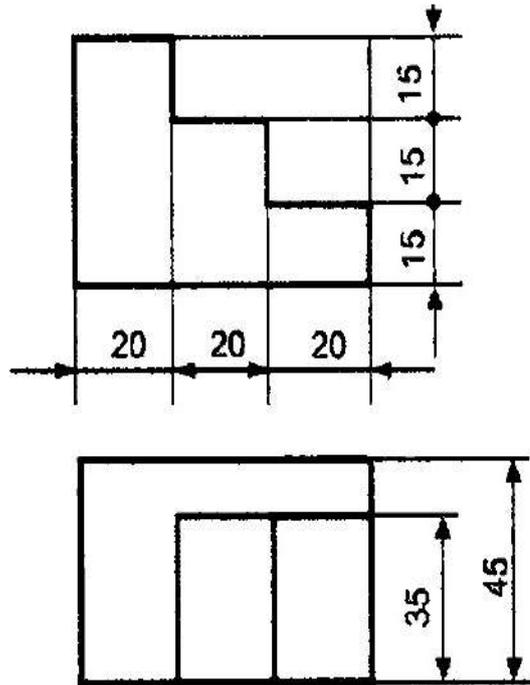


Figure 7

[12 + 8 = 20]