

PART I (STATISTICS / MATHEMATICS STREAM)

**ATTENTION: ANSWER A TOTAL OF SIX QUESTIONS TAKING
AT LEAST TWO FROM EACH GROUP - S1 AND S2.**

GROUP S1: Statistics

1. (a) Let X_1, \dots, X_n be a random sample from $N(\mu, \sigma^2)$ with unknown μ and known σ^2 . Find the UMVUE of μ^3 . State clearly any result that you may use.
- (b) A random sample of size n is drawn without replacement from a population of unknown size N , having N_1 (known) members of a specified type. If x is the number of individuals of this type in the sample, show that the MLE of N is approximately $\frac{nN_1}{x}$.

[10+10 = 20]

2. (a) What are the variables, influencing the response in an experiment, called?
- (b) What is the name of the phenomenon where the difference in response between the levels of one factor is not the same at all levels of the other factors?
- (c) What is the name of the design of experiments
- (i) which systematically allows blocking in two directions?
 - (ii) in which all possible combinations of the levels of the factors are investigated?
- (d) Which basic principle of experimental design makes the assumption that *the errors are independently distributed random variables* valid?
- (e) Consider the following fixed effects linear model under the usual Gauss-Markoff setup for the data collected from an experiment involving two treatments:

$$E(y_i) = \begin{cases} \mu + \beta_1, & j = 1,3,4 \\ \mu + \beta_2, & j = 2,5 \end{cases}$$

where β_i is the effect of the i th treatment, $i = 1,2$ and μ is the overall mean.

- (i) Name the design of experiments which might have been used to collect the above data.
- (ii) How will you compute the sum of squares due to treatment? [No derivation is required, just write the expression.] What is its degrees of freedom?
- (iii) Derive the expected value of the above sum of squares.

$$[1 \times 5 + 2 + (3 + 1) + 9 = 20]$$

3. It is believed that working in shipbuilding industry and smoking are two important causes of occurrence of lung cancer. In order to verify these beliefs several people are studied and the following are observed:

- 11 people who did not smoke but worked in shipbuilding had lung cancer
- 35 people who did not smoke but worked in shipbuilding did not have lung cancer
- 50 people who did not smoke and did not work in shipbuilding had lung cancer
- 203 people who did not smoke and did not work in shipbuilding did not have lung cancer
- 84 people who smoked and worked in shipbuilding had lung cancer
- 45 people who smoked and worked in shipbuilding did not have lung cancer
- 313 people who smoked but did not work in shipbuilding had lung cancer
- 270 people who smoked but did not work in shipbuilding did not have lung cancer

- (a) Present this data suitably in a tabular structure.

- (b) Do you think working in shipbuilding is likely to lead to lung cancer irrespective of whether a person smokes or not? Explain. (You may argue probabilistically without giving any formal test procedure).
- (c) Do you see any interaction effect of habit of smoking / not smoking and working / not working in shipbuilding on the occurrence of lung cancer? Justify.
- [10 + 5 + 5 = 20]
4. (a) Suppose we have n observations x_1, x_2, \dots, x_n . Let $\bar{x}_i = (\sum_{j=1}^i x_j)/i$, $i = 1, 2, \dots, n$ and $s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}_n)^2$. Show that $ns^2 = \sum_{i=2}^n \frac{i}{i-1} (x_i - \bar{x}_i)^2$.
- (b) Consider the polynomial model:
- $$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \epsilon_i, \quad i = 1, \dots, n,$$
- where ϵ_i 's are independent and identically distributed random variables with mean 0. Suppose that $n = 12$ and
- $$X_i = \begin{cases} -1, & i = 1, \dots, 4 \\ 0, & i = 5, \dots, 8 \\ 1, & i = 9, \dots, 12. \end{cases}$$
- (i) If the model is written as $Y = X\beta + \epsilon$, write down the X matrix.
- (ii) Are $\beta_0 - \beta_1$ and $\beta_0 + \beta_1 + \beta_2 + \beta_3$ estimable? Justify your answers.
- [10 + (4 + (3+3)) = 20]
5. (a) Let p be the probability that a coin will turn up head in a single toss. In order to test $H_0 : p = \frac{1}{2}$ against $H_1 : p = \frac{3}{4}$, the coin is tossed 5 times and H_0 is rejected if more than 3 heads are obtained. Find the probability of type I error and power of the test.
- (b) The principle of ordinary least squares is used to estimate the regression coefficients in simple linear regression studies of

y on x , and x on y , based on a random paired sample $\{(x_i, y_i), i = 1, \dots, n\}$. Let $e_{y \cdot x}$ represent the residual of y when y is regressed on x . Similarly, $e_{x \cdot y}$ represents the residual of x when x is regressed on y . If r ($\neq 0, \pm 1$) is the sample correlation coefficient then

- (i) find the correlation coefficient between x and $e_{y \cdot x}$, and interpret $e_{y \cdot x}$ based on the value so obtained, and
- (ii) compute the correlation coefficient between $e_{x \cdot y}$ and $e_{y \cdot x}$.

$$[(5+5) + ((3+1) + 6) = 20]$$

GROUP S2: Probability

- 6. (a) A and B throw alternatively a pair of dice in that order. The score in a throw is the sum of the outcomes of the two dice. A wins if he scores exactly 6 points before B gets exactly 7 points, and vice versa. If A starts the game what is the probability that he wins?
- (b) Let Y_1, Y_2 and Y_3 be the order statistics of X_1, X_2 and X_3 , where X_i 's are independent and uniformly distributed between 0 and 1. Find the distribution of $Z = Y_3 - Y_1$.

$$[10 + 10 = 20]$$

- 7. (a) The market for laundry detergent in a certain place deals with three brands, namely A, B and C. The customers show some brand loyalty, but tend to switch brands at a predictable rate from week to week. The proportion of buyers who switch from one brand of this week to another brand in the next week is given in the following table:

| | | Next Week | | |
|--------------|---|-----------|-----|-----|
| | | A | B | C |
| Current Week | A | 0.4 | 0.4 | 0.2 |
| | B | 0.5 | 0.4 | 0.1 |
| | C | 0.3 | 0.2 | 0.5 |

Thus the transition probability that a person who buys B in the current week will buy A in the next week is 0.5, and similarly other transitions are interpreted. Assuming that the probability of transition from brand to brand is not affected by the past history of brand transition, find the *steady state* market share of the brands.

- (b) The random variables X and Y have joint probability density function:

$$f(x, y) = ce^{-8x^2 - 6xy - 18y^2}, \quad -\infty < x, y < \infty.$$

Find the value of c and the correlation coefficient between X and Y .

[8 + 12 = 20]

8. (a) A car-wash facility has two service stations, S1 and S2, with service times distributed as exponential with rates δ_1 and δ_2 respectively. An entering car has to visit both the stations in that order. On completion of service at station S1, the car proceeds to S2 provided that S2 is unoccupied. Otherwise, it is made to wait at S1, blocking the entrance of any newly arriving car. The car leaves the facility immediately after its service at S2 is completed. When you arrive at the facility, you find a car being served at S1, and S2 is busy servicing another car. Compute your expected waiting time before you exit from the wash facility.
- (b) Let X and Y be independent random variables each having the standard normal distribution. Consider the circle centered at the origin and passing through the point (X, Y) in the xy -plane. What is the probability density function of the area of the circle? What is the expected value of this area?

[10 + (8+2) = 20]

9. (a) Let X be the outcome when a die is rolled once. Suppose that the probability of a particular outcome $X = x$ is proportional to x ($x = 1, 2, \dots, 6$). At first this die is rolled and its outcome X is noted. Next a fair coin is tossed X times and the number of heads so obtained is denoted by Y . Compute the followings:
- (i) $P[Y = 4]$
 - (ii) $P[X = 5 | Y = 4]$
 - (iii) $E[Y]$
 - (iv) $E[XY]$
- (b) Processing time of a job on the machine has Gamma distribution with shape parameter $\alpha = 4$ and scale (rate) parameter $\lambda = 2$ per hour. Use the central limit theorem to find the approximate probability that the machine completes 100 such jobs in 200 hours.

$$[(4 + 2 + 3 + 3) + 8 = 20]$$

10. (a) Suppose that Y is a discrete random variable defined on the sample space $\{0, 1, 2, \dots\}$. Let the conditional distribution of X given $Y = y$ be binomial with size y and probability p . Show that if Y has Poisson distribution with mean λ , then the marginal distribution of X is Poisson with mean λp .
- (b) In a coin tossing experiment, the probability of head is p and that of tail is q , where $0 < p < 1$ and $p + q = 1$. Suppose that the coin is tossed three times. Let X be the number of heads in the first two tosses and Y be the number of heads in the last two tosses. What is the joint probability distribution of (X, Y) ? Find the expectation of XY .
- (c) Let X be an absolutely continuous random variable symmetric about 0. Assume that X has finite second moment. Define $Y = |X|$. Show that X and Y are uncorrelated but are not independent.

$$[7 + (5 + 2) + 6 = 20]$$

PART II (ENGINEERING STREAM)

**ATTENTION: ANSWER A TOTAL OF SIX QUESTIONS TAKING
AT LEAST TWO FROM EACH GROUP - E1 AND E2.**

GROUP E1: Mathematics

1. (a) If $z + \frac{1}{z} = 2 \cos \theta$, where z is a complex number, show that

$$\left| \frac{z^{2n} - 1}{z^{2n} + 1} \right| = |\tan n\theta|$$

- (b) If $p \neq 0$, $q \neq 0$ and

$$\det \begin{bmatrix} p & q & p\alpha + q \\ q & r & q\alpha + r \\ p\alpha + q & q\alpha + r & 0 \end{bmatrix} = 0,$$

then prove that either of the following statements is true:

- (i) p , q and r are in geometric progression,
- (ii) α is a root of the equation $px^2 + 2qx + r = 0$.

[8+12=20]

2. (a) Show that the sum of all positive integers up to 1000 which are divisible by 5 and not divisible by 2 is 50000.
- (b) Suppose that a geometric progression (GP) $1, q, q^2, \dots$ (where $|q| < 1$) is to be constructed such that every term of this GP is a constant multiple (say k) of sum of all the subsequent terms. For what values of k is this problem solvable?
- (c) A rectangle ABCD is inscribed in a circle. Let PQ be the diameter parallel to the side AB of the rectangle. Suppose that the measure of the angle BPC is 30° . Show that the ratio of the area of the rectangle and the area of the circle is $\sqrt{3}/\pi$.

[4+6+10=20]

3. (a) Let $I = \int_{-\pi}^{\pi} \frac{\sin^2 x}{1+a^x} dx$, $a > 0$. Find the value of I .
- (b) A circle, having center at (2, 3) and radius 6, crosses y-axis at the points P and Q. The straight line with equation $x = 1$ intersects the radii CP and CQ at points R and S respectively. Find the area of the trapezium PQSR.
- [10+10=20]
4. (a) Consider a parallelogram ABCD. Suppose that P is a point on the side AD so that $AP : AD = x : y$. Let Q be the intersection point of AC and PB. Show that $AQ : AC = x : x + y$.
- (b) Find the derivative of $f(x)$ at $x = 0$, where
- $$f(x) = \frac{\sqrt{1+2x} \cdot \sqrt[4]{1+4x} \cdot \sqrt[6]{1+6x} \cdots \sqrt[100]{1+100x}}{\sqrt[3]{1+3x} \cdot \sqrt[5]{1+5x} \cdot \sqrt[7]{1+7x} \cdots \sqrt[101]{1+101x}}$$
- (c) A spherical balloon is being inflated so that its volume increases uniformly at the rate of $40 \text{ cm}^3/\text{minute}$. What is the rate at which the surface area is increasing when the radius of the balloon is 8 cm?
- [5+7+8=20]

GROUP E2: Engineering & Technology

Engineering Mechanics and Thermodynamics

5. (a) A block '1' of weight W_1 rests on a horizontal surface and supports on the top of it a block '2' of weight W_2 . The block '2' is attached to a vertical wall by the inclined string AB as shown in Fig. 1. Find the magnitude of the horizontal force P , applied to the lower block '1', which will be necessary to cause slipping to impend. The coefficient of static friction for all contiguous surfaces is $\mu = 0.3$.

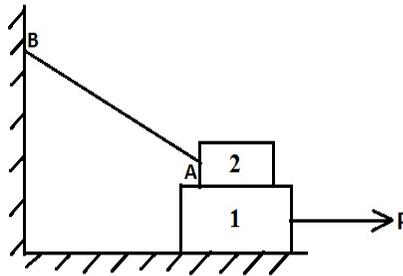


Fig. 1

- (b) Three identical cylinders, each weighing W , are stacked on smooth inclined surfaces as shown in Fig. 2. Determine the smallest angle θ to prevent the stack from collapsing.

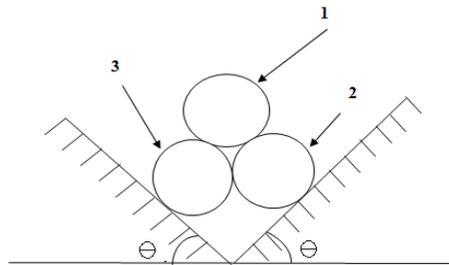


Fig. 2

[10+10=20]

6. (a) Using an engine of 40% thermal efficiency to drive a refrigerator having COP of 5, what will be the heat input into the engine for removal of 2 MJ from the cold body by the refrigerator? If this system is used as a heat pump, how many MJ of heat will be available for heating for each MJ of heat input to the engine?
- (b) An engine cylinder has a piston area of 0.12 m^2 and contains a gas at a pressure of 1.5 MPa. The gas expands according to a process which is represented by a straight line on the P - V diagram. The final pressure of the gas is 0.15 MPa. Calculate the work done by the gas on the piston if the stroke is 0.30 m.

- (c) The efficiency of a Carnot heat engine can be increased either by increasing the source temperature while the sink temperature is held constant or by decreasing the sink temperature while the source temperature is held constant. Which of the above two possibilities is more effective?

[(3+3)+4+10=20]

7. (a) An engine shaft running at 120 rpm is required to drive a machine shaft by means of a belt. The pulley on the engine shaft is of 2 m diameter and that of the machine shaft is of 1 m diameter. If the belt thickness is 5 mm, find the speed of the machine shaft when (i) there is no slip, and (ii) there is a slip of 3%.
- (b) From the first law of thermodynamics, show that the change of entropy of a perfect gas (ΔS) having mass m is $\Delta S = mC_p \ln \frac{T_2}{T_1}$, where C_p is the specific heat at constant pressure, T_1 is the initial temperature of the gas and T_2 is the final temperature of the gas in K.

[(5+5)+10=20]

Electrical and Electronics Engineering

8. (a) A three-phase star-connected synchronous generator driven at 750 rpm is required to generate a line-to-line voltage of 440 V at 50 Hz on open circuit. The stator is wound with 2 slots per pole per phase and each coil has 4 turns. Calculate the useful flux per pole. [Assume a full-pitch coil. $\sin 15^\circ = 0.26$].
- (b) A three-phase, 6-pole, 50 Hz induction motor develops the maximum torque at 940 rpm. The rotor resistance per phase is 0.1 Ω . Calculate the value of the standstill rotor reactance.
- (c) A dc shunt machine has an armature resistance of 0.5 Ω and a field-circuit resistance of 750 Ω . When run as a motor on no-load at 500 V applied terminal voltage, the line current

was 3 A. Estimate the efficiency of the machine when it operates as a generator with an output of 20 kW at 500 V, the field circuit resistance remaining unchanged.

[8+5+7=20]

9. (a) Calculate the average and rms values of the current $i(t) = 4 + 10 \sin \omega t$ A.
- (b) Show that a full adder can be constructed with two half adders and an OR gate using necessary truth tables, equations and circuit diagrams.
- (c) A cube is formed by joining equal wires, each of resistance 1Ω . The cube is shown in Fig. 3. Calculate the equivalent resistance between the points A and B.

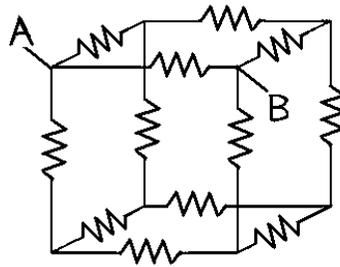


Fig. 3

[(2+3)+10+5=20]

Engineering Drawing

10. (a) Sketch the sectional front view, top view and right side view of the object shown in Fig. 4.

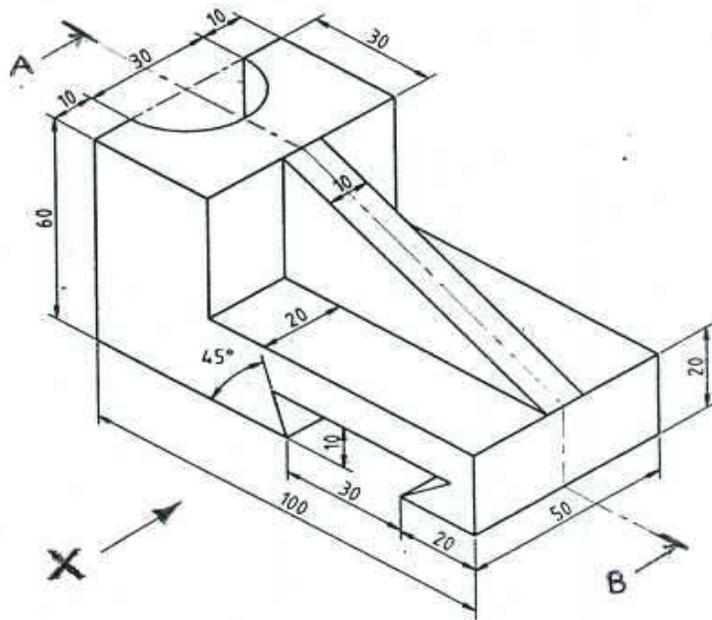


Fig. 4

(b) Sketch the three views of the object shown in Fig. 5.

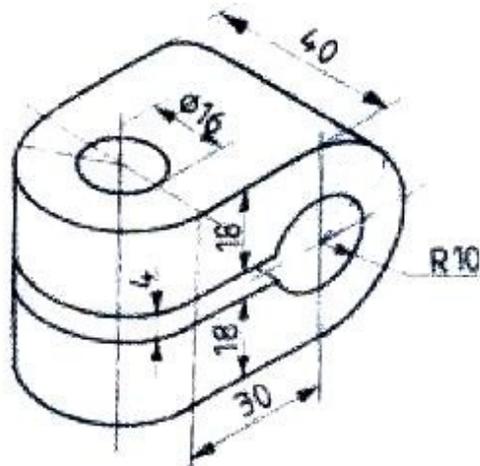


Fig. 5

[12+8=20]