Group A: CS Group

Answer Question C1 and any 5 questions from C2 to C11.

C1. Given an array $A = (a_0, a_1, \ldots, a_{n-1})$ of integers, your task is to determine the maximum possible sum

$$\text{maxsum}(A) = \max_{0 \le s \le t \le n-1} a_s + a_{s+1} + \cdots + a_{t-1} + a_t.$$

For example, if $A = (-2, 1, -3, 4, -1, 2, 1, -5, 4)$, then $\text{maxsum}(A) = 6$, that is attained for $s = 3$ and $t = 6$.

(a) Write a function that computes $\text{maxsum}(A)$ given an array A, as a pseudocode or in a programming language of your choice. *Note that you will be awarded 6 marks for writing a correct code and 6 marks will be given based on the efficiency of your program.*

(b) Argue why your code is correct, and determine the worst-case computational complexity of your function with respect to the length of the array.

[(6 + 6) + (4 + 4) = 20]

C2. (a) Let $b_n b_{n-1} \cdots b_1$ be the decimal representation of an n digit number m. Let $b_n b_{n-1} \cdots b_2$ be the integer a obtained from m by stripping off the unit's digit b_1. Then, show that m is divisible by 7 if and only if $a - 2b_1$ is divisible by 7.

(b) Prove that:

$$1 + \binom{1001}{1} + \binom{1002}{2} + \cdots + \binom{2023}{1023} + \binom{2024}{1024} = \binom{2025}{1024}.$$

[10 + 6 = 16]

C3. Suppose $X = (x_1, x_2, \ldots, x_n)$ is an array of numbers (not necessarily integers) sorted in the ascending order, and $Y = (y_1, y_2, \ldots, y_n)$ is the array constructed as $y_i = f(x_i)$ for $i = 1, 2, \ldots, n$, where

$$f(x) = (x+1)(x-1)(x-3).$$

(a) Describe an algorithm that sorts Y using $O(n)$ comparisons.

(b) Provide justification for the correctness and the number of comparisons used by your algorithm.

[10 + 6 = 16]

C4. Let R be the root of a balanced binary search tree having n nodes, where each node contains an integer, a pointer to its left child and a pointer to its right child. Note that insertion and deletion of nodes in such a tree can be done in $O(\log n)$ time in the worst case.

(a) Describe a procedure previous(R, x) that, given the root R and any integer x, outputs the node containing the largest integer smaller than x (or, reports if no such node exists). The worst case time complexity of the procedure should be $O(\log n)$. Provide a justification for the complexity of your procedure.

(b) Describe a procedure findLargest(R, k) that, given R and an integer $k \leq n$, outputs the k-th largest integer present in the tree in $O(\log n)$ time. For this purpose, you may store some extra information in each of the nodes, but ensure that the insertion and deletion of nodes in the tree can still be done in $O(\log n)$ time in the worst case.

[8 + 8 = 16]

C5. For a graph $G = (V, E)$ and a subset $X \subseteq V$ of the vertices, the induced subgraph $G[X]$ is defined as $G[X] = (X, E')$, where $E' = \{uv \in E \mid u, v \in X\}$.

Suppose $G = (V, E)$ is a connected undirected graph on two or more vertices and $T = (V, F)$ is a spanning tree of G. Let $S \subset V$ be a non-empty proper subset of the vertices and define $A = \{uv \in E \mid u \in S, v \in V \setminus S\}$. Prove or disprove each of the following:

(a) $A \cap F \neq \varnothing$.

(b) If $G[S]$ has s connected components and $G[V \setminus S]$ has t connected components, then $A \cap F$ contains at least $s+t-1$ edges.

(c) If $A \subseteq F$, then either $T[S]$ or $T[V \setminus S]$ is disconnected.

$[4 + 8 + 4 = 16]$

C6. For an attribute set A with a set of underlying functional dependencies, the closure A^+ is defined as the set of attributes that can be derived from A. Given a relation R with attribute sets X, Y, Z, prove or disprove each the following statements.

(a) $(X \cup Y)^+ \cap (X \cup Z)^+ = X^+ \cup (Y \cap Z)^+$.

(b) $(X \cap Y)^+ \cup (X \cap Z)^+ = X^+ \cap (Y \cup Z)^+$.

$[8 + 8 = 16]$

C7. Using JK flip-flops and a minimal number of combinatorial logic gates, design a sequential circuit that takes a stream of bits as input and outputs 1 whenever three consecutive 0's appear in the input stream, and outputs 0 otherwise. Show the detailed steps of your design with suitable justifications.

[16]

C8. (a) Consider the following piece of a program for a hypothetical processor having two user registers `R1` and `R2`.

Instruction	Underlying Operation	Instruction Size (in words)
`MOVE R2, 1000;`	R2 = Mem[1000]	2
`ADD R1, R2;`	R1 = R1 + R2	1
`MOVE 2000, R1;`	Mem[2000] = R1	2
`MOVE R1, R2;`	R1 = R2	1

Assume that the clock cycles required for various operations are given as follows:

- Data transfer between register and memory: 4 clock cycles,
- Data transfer between two registers: 1 clock cycle,
- `ADD` with both operands in register: 2 clock cycles, and
- Instruction fetch and decode: 3 clock cycles per word.

How many clock cycles are required to execute the above program segment?

(b) A hypothetical computer system has two levels (L1 and L2) cache where L1 and L2 has their hit rates as 94% and 80%, respectively, and miss penalties as 10 and 60 cycles, respectively. Let the base CPI of this computer system be 1.5 when there is no cache miss. If on average, there are 1.2 memory accesses per instruction, then what is the effective CPI once cache miss occurs?

[6 + 10 = 16]

C9. (a) Let L be a language over an alphabet $\{a, b\}$ such that the empty string does not belong to L and the first and the last letters of every string in L are the same. Draw a DFA to accept the language L. Argue whether your DFA is the minimal one accepting L.

(b) Prove that the language $L = \{a^m b^n \mid m \neq n\}$ is not regular. You may use the fact that the language $\{a^n b^n \mid n \geq 0\}$ is not regular.

$[(6 + 4) + 6 = 16]$

C10. Suppose the following code is compiled, and run for different values of `N`.

```
int main() {
    for (int i = 1; i <= N; i++) {
        fprintf(stderr, "OPERATING\n");
        fork();
        fork();
    }
    fprintf(stderr, "SYSTEMS\n");
    return 0;
}
```

Assume that the `fork()` system call always succeeds.

(a) How many times will the strings `OPERATING` and `SYSTEMS` be printed if the value of `N` is (i) 1, (ii) 2?

(b) For a given `N`, let $I_{\texttt{N}}$ and $J_{\texttt{N}}$ represent, respectively, the number of times the strings `OPERATING` and `SYSTEMS` are printed. Derive recurrence relations for $I_{\texttt{N}}$ and $J_{\texttt{N}}$.

$[(2 + 4) + 10 = 16]$

C11. Consider that node X is sending packets to node Y. The path from X to Y has a bandwidth of $B_{XY} = 5000$ bytes/second and a propagation delay of $P_{XY} = 100$ milliseconds. The reverse path from Y to X has a bandwidth of $B_{YX} = 9500$ bytes/second and a propagation delay of $P_{YX} = 40$ milliseconds. Suppose the size of the data packets is $S_D = 500$ bytes and the size of the acknowledgement packets is $S_A = 95$ bytes. Ignore the size of the header in data packets. Moreover, assume that no data packets or acknowledgement packets are lost during transit and there is no queuing delay.

(a) Compute the throughput in bytes/second that the node X can achieve in transmitting to node Y using *Stop-and-Wait* protocol.

(b) Suppose instead of Stop-and-Wait, node X uses *Sliding Window* protocol.

(i) Compute the size of the window W, in terms of number of data packets, that node X must use in order to transfer its data at the maximum possible rate. Determine the actual rate $T_{\max}$ that X achieves while utilizing the window of size W.

(ii) If the bandwidth B_{YX} of the path from Y to X drops to 125 bytes/second, keeping B_{XY}, P_{XY}, P_{YX}, S_D and S_A unchanged, can X still achieve the transfer rate $T_{\max}$? If so, what is the new window size W? If not, what is the new limit on $T_{\max}$ that X can achieve?

$[4 + ((4 + 2) + (4 + 2)) = 16]$

Group B: Non-CS Group

Answer Question N1 and any 5 questions from N2 to N13.

N1. (a) Consider the following function `job()`, which takes two positive integers `x` and `y`, and returns another integer.

```
int job(int x, int y) {
  if (x==y) return x;
  else if (x > y) return job(x-y, y);
  else return job(x, y-x);
}
```

What will be the value of `job(18,24)`? Mathematically, what does the function `job(x,y)` compute for two given positive integers `x` and `y`?

(b) A positive integer is called *automorphic* if its square ends in the same digits as the number itself. For example, 25 is an automorphic number as its square 625 ends with 25, whereas 16 is not automorphic because its square 256 does not end with 16. Fill in the blanks in the pseudocode provided below to write the function `isAutomorphic`, which determines whether a given number is automorphic. Note that `a / b` and `a % b` compute the quotient and remainder, respectively, when `a` is divided by `b`.

```
boolean isAutomorphic(int x) {
  int a := _________;
  int c := x;
  while (c > 0) {
    a := a * 10;
    c := c / ________;
  }
  if ((x*x) % a == ________) return TRUE;
  else return FALSE;
}
```

$[(5 + 6) + 9 = 20]$

N2. (a) Let $b_n b_{n-1} \cdots b_1$ be the decimal representation of an n digit number m. Let $b_n b_{n-1} \cdots b_2$ be the integer a obtained from m by stripping off the unit's digit b_1. Then, show that m is divisible by 7 if and only if $a - 2b_1$ is divisible by 7.

(b) Prove that:

$$1 + \binom{1001}{1} + \binom{1002}{2} + \cdots + \binom{2023}{1023} + \binom{2024}{1024} = \binom{2025}{1024}.$$

[10 + 6 = 16]

N3. Suppose $X = (x_1, x_2, \ldots, x_n)$ is an array of numbers (not necessarily integers) sorted in the ascending order, and $Y = (y_1, y_2, \ldots, y_n)$ is the array constructed as $y_i = f(x_i)$ for $i = 1, 2, \ldots, n$, where

$$f(x) = (x+1)(x-1)(x-3).$$

(a) Describe an algorithm to sort Y using $O(n)$ comparisons.

(b) Provide justification for the correctness and the number of comparisons used by your algorithm.

[10 + 6 = 16]

N4. (a) Let A be an $n \times n$ matrix and let I denote the $n \times n$ identity matrix. Show that, if $A^3 = \mathbf{0}$, then $(I - A)$ is invertible.

(b) Let V be an n-dimensional vector space over $\mathbb{R}$, and let $F : V \to V$ be a linear transformation such that $F \circ F = I$, where $I : V \to V$ is the identify transformation. Prove that $V = \operatorname{Im}(I + F) \oplus \operatorname{Im}(I - F)$, where $\oplus$ denotes the direct sum of two subspaces and $\operatorname{Im}(T)$ denotes the image of a transformation T.

[8 + 8 = 16]

N5. Let $f : \mathbb{R} \to (0, \infty)$ be a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

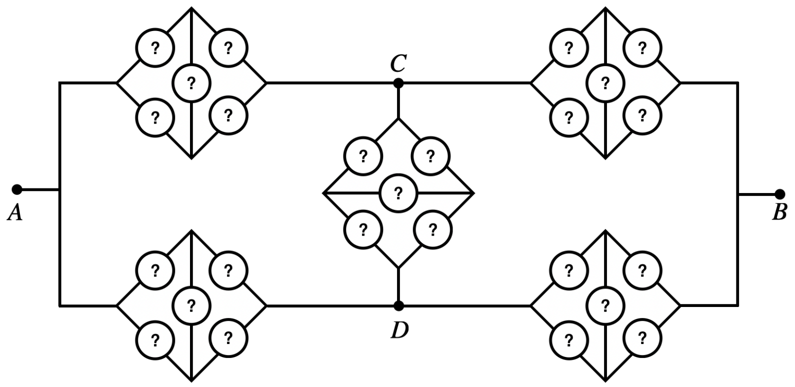
(a) Prove that if f is continuous at 0, then f is continuous at every $x \in \mathbb{R}$.

(b) Prove or disprove: If f is differentiable at 0, then f is differentiable at every $x \in \mathbb{R}$.

(c) Determine all such functions f which are differentiable everywhere on $\mathbb{R}$.

$[6 + 6 + 4 = 16]$

N6. The map below shows all the routes connecting the locations A, B, C and D. However, there are many bridges, each indicated with a ? mark. Each bridge is open with probability p (and closed with probability $1 - p$), independent of the state of all other bridges. When a bridge is closed, the corresponding segment of the route is also closed.



(a) Determine the probability that there is an open route from A to C that *does not* pass through D.

(b) Subsequently, determine the probability that there is an open route from A to B.

$[8 + 8 = 16]$

N7. For a graph $G = (V, E)$ and a subset $X \subseteq V$ of the vertices, the induced subgraph $G[X]$ is defined as $G[X] = (X, E')$, where $E' = \{uv \in E \mid u, v \in X\}$.

Suppose $G = (V, E)$ is a connected undirected graph on two or more vertices and $T = (V, F)$ is a spanning tree of G. Let $S \subset V$ be a non-empty proper subset of the vertices and define $A = \{uv \in E \mid u \in S, v \in V \setminus S\}$. Prove or disprove each of the following:

(a) $A \cap F \neq \varnothing$.

(b) If $G[S]$ has s connected components and $G[V \setminus S]$ has t connected components, then $A \cap F$ contains at least $s + t - 1$ edges.

(c) If $A \subseteq F$, then either $T[S]$ or $T[V \setminus S]$ is disconnected.

$[4 + 8 + 4 = 16]$

N8. Let $p(x) = (x-1)(x-2)\cdots(x-n+1)(x-n)-1$ be a polynomial where $n \geq 1$. Suppose $p(x) = f(x)g(x)$ where f and g are both polynomials with integer coefficients.

(a) Show that $f(k) = -g(k)$ for all $k = 1, 2, \ldots, n$.

(b) Show that either $f(x) = -g(x)$ for all $x \in \mathbb{R}$, or one of f and g is a constant polynomial.

(c) Subsequently, prove that $p(x)$ cannot be written as a product of two non-constant polynomials with integer coefficients.

$[2 + 8 + 6 = 16]$

N9. (a) Let L be a language over an alphabet $\{a, b\}$ such that the empty string does not belong to L and the first and the last letters of every string in L are the same. Draw a DFA to accept the language L. Argue whether your DFA is the minimal one accepting L.

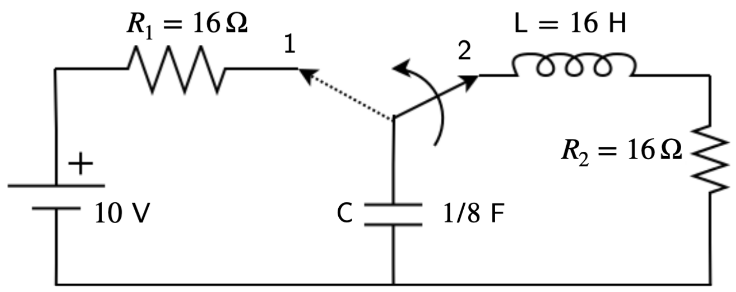
(b) Prove that the language $L = \{a^m b^n \mid m \neq n\}$ is not regular. You may use the fact that the language $\{a^n b^n \mid n \geq 0\}$ is not regular.

$[(6 + 4) + 6 = 16]$

N10. Using JK flip-flops and a minimal number of combinatorial logic gates, design a sequential circuit that takes a stream of bits as input and outputs 1 whenever three consecutive 0's appear in the input stream, and outputs 0 otherwise. Show the detailed steps of your design with suitable justifications.

[16]

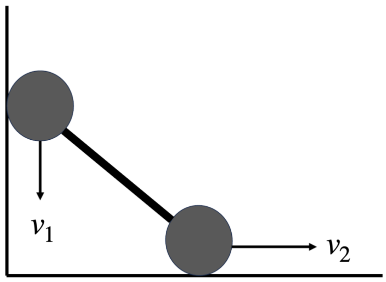
N11. The switch S shown in the figure is initially at position '2' and is moved to position '1' at $t = 0$ second. Then, at $t = 2$ seconds, the switch S is again moved to position '2'. Find the voltage across the resistor R_2 for $t > 2$ seconds. You can express the final answer in terms of sine, cosine, and exponential functions.



[16]

N12. A dumbbell consisting of two balls, both of mass m, connected by a massless rod of length L rests on a frictionless floor against a frictionless wall until it begins to slide down the wall, as in the figure below. Derive an expression for the speed of the balls at the moment when their speeds are equal, in terms of L and g, where g denotes the gravitational acceleration.

Treat the dumbbell as two point masses separated by a massless rod. Also assume that the initial state of the dumbbell is vertical and the image shows the dumbbell some time after it has fallen.



[16]

N13. A thermally insulated cylinder, closed at both ends, is fitted with a frictionless heat-conducting piston that divides the cylinder into two parts. Initially, the piston is clamped in the center, with air of volume V_0 at temperature T_1 and pressure p_1 on one side, and the same volume of air at temperature T_2 and pressure p_2 on the other side. The piston is then released, allowing the system to reach equilibrium in pressure and temperature, with the piston in a new position. Assume that the molar specific heat of air at constant volume is $C_v = \frac{3}{2}R$.

(a) Derive expressions for the final pressure and final temperature of the air in the cylinder.

(b) Derive an expression for the total increase in entropy.

[10 + 6 = 16]