PCB Model Question Paper

Disclaimer: The following model question paper is provided solely for illustrative purposes to demonstrate the structure and format of the actual examination. Please note that the difficulty level and topic-wise distribution of questions in the final exam may vary.

The question paper is divided into two groups: *CS Group* and *Non-CS Group*. Please choose **ANY ONE** group and answer the questions from that group only, according to the instructions.

CS Group

Answer Question C1 (20 marks) and any 5 from the other questions ($5 \times 16 = 80$ marks).

C1. Consider a reservoir, fitted with a set of taps and drains. The partial pseudocode given below takes as input:

L	capacity of the reservoir in litres
T	number of taps attached to the reservoir
D	number of drains attached to the reservoir
t_i	time in minutes that it takes to fill an initially empty reservoir if all drains are stopped, and only the i-th tap is opened ($1 \leq i \leq T$)
d_j	time in minutes that it takes to drain an initially full reservoir if all taps are shut off, and only the j-th drain is opened ($1 \leq j \leq D$)

Fill in the blanks below, so that the statements printed by the pseudocode are correct. Do not copy the pseudocode; simply write the serial number of each blank (I, II, ..., VI), and your corresponding answer.

Pseudocode:

Initially, set variables X and Y to 0.

for i in $\{1, 2, 3, \ldots, T\}$ $\quad X \longleftarrow X +$ ____(I)____.

for j in $\{1, 2, 3, \ldots, D\}$ $\quad Y \longleftarrow Y +$ ____(II)____.

If X ____(III)____ Y
then print "*A partially empty reservoir never fills up if all taps and drains are kept open.*"
else print "*An initially empty reservoir fills up in* ____(IV)____ *minutes if all taps and drains are kept open.*"

If X ____(V)____ Y
then print "*A partially full reservoir never becomes empty if all taps and drains are kept open.*"
else print "*An initially full reservoir becomes empty in* ____(VI)____ *minutes if all taps and drains are kept open.*"

C2. You are given 68 identical looking balls, each with a distinct weight. You are given a common balance using which you can compare weights of any pair of balls with a single measurement. Describe how you would identify both the heaviest and the lightest balls by using at most 100 measurements in total.

C3. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots b_m\}$ be two sorted arrays of n and m numbers, respectively. Devise an algorithm to count the number of pairs $(i, j), 1 \leq i \leq n, 1 \leq j \leq m$, so that $a_i > b_j$. The efficiency of your algorithm will be counted in terms of the number of comparisons you perform between the elements of A and B. For full credit, your algorithm should perform $\mathcal{O}(n + m)$ comparisons.

C4. Let k be a positive integer, and $n = 2^k - 1$. Let $X = \{1, 2, \ldots, n\}$ be arranged as a min-heap stored in a complete binary tree, T.

(a) What can be the smallest possible value that can be stored at a leaf node of T? Prove your result.

(b) Define the level of the root of the min-heap T to be 1. What can be the maximum possible value that can be stored at level 2 of T? Prove your result.

C5. For any undirected connected graph G, let $\chi(G)$ be the minimum number of colours needed to colour all the vertices of G in such a way that no two adjacent vertices have the same colour. Prove that for every vertex v of G, there exists a path of length $\chi(G) - 1$ starting from v. A path of length k is a sequence of $k + 1$ distinct vertices $v_1, v_2, \ldots, v_{k+1}$ such that (v_i, v_{i+1}) is an edge for $1 \leq i \leq k$.

C6. A 32-bit pattern $(p_{31}, \ldots, p_0)$ is interpreted as a floating point number as follows.

- The *sign* bit is $s = p_{31}$. If $s = 0$, the number is considered positive, otherwise it is considered negative.
- The 8-bit part $(p_{30}, \ldots, p_{23})$ is the *exponent* biased by 127. That is, if these bits represent an integer e, then the corresponding term would be considered as 2^{e-127}. The all-zero and the all-one bit patterns are not considered valid for these eight bits.
- The *mantissa* part $m = (p_{22}, \ldots, p_0)$ would be interpreted as $1.m$, where the leftmost 1 is implicitly understood.

With the above understanding, the decimal value of the 32-bit pattern becomes $(-1)^s \times 2^{e-127} \times 1.m$. For example, consider the 32-bit pattern (**1** 1011 0110 **011 0000 0000 0000 0000 0000**), which is actually $(-1)^1 \times 2^{10110110-01111111} \times 1.011$ as represented with bits, the same as -1.375×2^{55} in decimal.

(a) Suggest how the decimal zero can be represented in the above format.

(b) Describe the largest positive value in this format.

(c) Describe the smallest non-zero positive value in this format.

(d) Given two such bit patterns $(x_{31}, \ldots, x_0), (y_{31}, \ldots, y_0)$ in the above format, construct the digital circuit for their addition.

C7. Consider the alphabet $\Sigma = \{0, 1\}$.

(a) Design a finite automaton accepting the language $A = \{0^k w 0^k \mid k \geq 1 \text{ and } w \in \Sigma^*\}$.

(b) Show that no finite automaton accepts the language $B = \{0^k 1 w 0^k \mid k \geq 1 \text{ and } w \in \Sigma^*\}$.

C8. Let $P_1, P_2, \ldots, P_{20}$ be a set of 20 cooperating processes, all of which have access to a shared integer `X`. For $i \in \{1, 2, \ldots, 20\}$, the process P_i calls `do_something(i)`, the code for which is given below.

```
shared int X = 0;
void do_something(int i)
{
  printf("%d %d", i, X);
  X = X + i;
}
```

(a) What is the **minimum** possible final value of `X`? Justify your answer.

(b) Your task is to synchronise the execution of the above processes using semaphores so that the combined output of all 20 processes is as follows.

`1 1   2 3   3 6 ... 20 210`

i. Show how you would declare and initialise the necessary semaphore(s). Follow the syntax used in the examples given below.

```
shared semaphore S = 1;
shared semaphore mutex[m] = {0, 0, ..., 0};
```

ii. Modify the body of the above function by inserting appropriate `wait()` and `signal()` system calls so that the messages are printed in the desired order.

C9. Design a synchronous counter that has an *Enable* input. When *Enable* $= 1$, it generates a binary sequence equivalent to $0, 3, 1, 2, 0, 3, 1, 2, \ldots$ in decimal. When *Enable* $= 0$, the counter remains in its current state. Use J-K flip flops only and a minimal number of AND-OR-NOT gates in your design. Argue why your gate count is minimal. Modify your circuit without changing the input-output behaviour, so that it uses universal gates only.

C10. Consider sending the 6-bit message 101000 using Hamming codes for single-bit error-correction.

(a) Calculate the codeword for this message. Use even parity.

(b) Invert the last bit of the codeword and then show how the recipient can correct this inverted bit.

(c) Invert the first bit of the codeword and then show how the recipient can correct this inverted bit.

C11. Suppose the relationship between customers and products are stored in a table `BOUGHT(CUST,PROD,QT)` such that there is a tuple $(c, p, n) \in$ `BOUGHT` if and only if the customer c has bought the product p and n is the total quantity of p bought by c.

Fill the blanks in the given SQL statement so that it outputs a table COMMON_PRODS, which consists of the number of common products bought by every pair of customers who have bought some common product.

Do not copy the statements; simply write the serial number of each blank (I, II, ..., VII), and your corresponding answer.

```
 select CUST1, CUST2,______(I)______ as NUM
from (select ______(II)______ as CUST1,
______(III)______ as CUST2 from ______(IV)______ as B1, ______(V)______
as B2 where ______(VI)______) as T group by ______(VII)______;
```

Example: For the table BOUGHT as shown in Table 1, the output table COMMON_PRODS should be as shown in Table 2.

BOUGHT		
CUST	PROD	QT
1	A	6
2	A	8
1	B	2
1	C	2
2	C	1
3	B	9
3	C	5
4	B	2

Table 1: BOUGHT

CUST1	CUST2	NUM
1	2	2
1	3	2
1	4	1
2	3	1
3	4	1

Table 2: COMMON_PRODS

Non-CS Group

Answer Question N1 (20 marks) and any 5 from the other questions ($5 \times 16 = 80$ marks).

N1. Consider a reservoir, fitted with a set of taps and drains. The partial pseudocode given below takes as input:

L	capacity of the reservoir in litres
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d_j	time in minutes that it takes to drain an initially full reservoir if all taps are shut off, and only the j-th drain is opened ($1 \leq j \leq D$)

Fill in the blanks below, so that the statements printed by the pseudocode are correct. Do not copy the pseudocode; simply write the serial number of each blank (I, II, ..., VI), and your corresponding answer.

Pseudocode:

Initially, set variables X and Y to 0.

for i in $\{1, 2, 3, \ldots, T\}$ $\quad X \longleftarrow X+$ ___(I)___ .

for j in $\{1, 2, 3, \ldots, D\}$ $\quad Y \longleftarrow Y+$ ___(II)___ .

If X ___(III)___ Y
then print "*A partially empty reservoir never fills up if all taps and drains are kept open.*"
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then print "*A partially full reservoir never becomes empty if all taps and drains are kept open.*"
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N2. You are given 68 identical looking balls, each with a distinct weight. You are given a common balance using which you can compare weights of any pair of balls with a single measurement. Describe how you would identify both the heaviest and the lightest balls by using at most 100 measurements in total.

N3. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots b_m\}$ be two sorted arrays of n and m numbers, respectively. Devise an algorithm to count the number of pairs $(i, j), 1 \leq i \leq n, 1 \leq j \leq m$, so that $a_i > b_j$. The efficiency of your algorithm will be counted in terms of the number of comparisons you perform between the elements of A and B. For full credit, your algorithm should perform $\mathcal{O}(n + m)$ comparisons.

N4. Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix such that for $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$,

(a) a_{ij}s are real valued and independent random variables;

(b) $\mathbb{E}[a_{ij}] = 0$ and $\mathrm{Var}(a_{ij}) = 1$.

Let $\mathbf{x} \in \mathbb{R}^n$. Show that $\mathbb{E}[\|\mathbf{A}\mathbf{x}\|^2] = m\|\mathbf{x}\|^2$. Note that $\|\mathbf{x}\|^2 = \mathbf{x}^{\mathbf{T}}\mathbf{x}$.

N5. For a polynomial of the form $a_0 + a_1 x + \cdots + a_m x^m$ having degree $m > 1$, it is known that the coefficients satisfy the relation $\dfrac{a_{i-1}}{a_{i-1} + a_i} = \dfrac{i}{n+1}$. Show that the polynomial has at least one real root with multiplicity greater than 1.

N6. A and B are playing a game which involves tossing a fair coin repeatedly. A wins if the sequence HHT appears before the sequence HTH. Compute the probability that A wins.

N7. For any undirected connected graph G, let $\chi(G)$ be the minimum number of colours needed to colour all the vertices of G in such a way that no two adjacent vertices have the same colour. Prove that for every vertex v of G, there exists a path of length $\chi(G) - 1$ starting from v. A path of length k is a sequence of $k + 1$ distinct vertices $v_1, v_2, \ldots, v_{k+1}$ such that (v_i, v_{i+1}) is an edge for $1 \leq i \leq k$.

N8. Consider the alphabet $\Sigma = \{0, 1\}$.

(a) Design a finite automaton accepting the language $A = \{0^k w 0^k \mid k \geq 1 \text{ and } w \in \Sigma^*\}$.

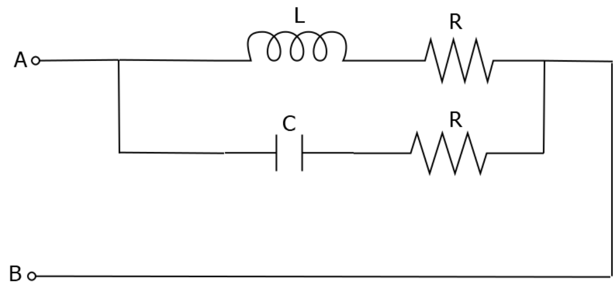
(b) Show that no finite automaton accepts the language $B = \{0^k 1 w 0^k \mid k \geq 1 \text{ and } w \in \Sigma^*\}$. [5+5]

N9. Let $f : (-1, 1) \to \mathbb{R}$ be twice differentiable such that $f(1/n) = 0$ for all $n \in \mathbb{N}$. Show that (i) $f'(0) = 0$, and (ii) $f''(0) = 0$.

N10. Design a synchronous counter that has an *Enable* input. When *Enable* $= 1$, it generates a binary sequence equivalent to $0, 3, 1, 2, 0, 3, 1, 2, \ldots$ in decimal. When *Enable* $= 0$, the counter remains in its current state. Use J-K flip flops only and a minimal number of AND-OR-NOT gates in your design. Argue why your gate count is minimal. Modify your circuit without changing the input-output behaviour, so that it uses universal gates only.

N11. An engine operates at 75% of the efficiency of a Carnot engine operating between two temperatures T_1 and T_2 ($T_1 > T_2$). This engine has a power output of 100 W and discharges heat at the rate of 300 J/s into the low-temperature reservoir with $T_2 = 27°$C. Compute the temperature T_1 of the high-temperature reservoir.

N12. In the circuit shown below, determine the resonant frequency and the resonant impedance between terminals A and B, when $R = 100\ k\Omega$, $C = 10\ \mu F$, and $L = 100\ mH$. Find the condition in terms of R, C and L, under which the impedance between A and B at resonance equals R.



N13. A small block starts sliding down an inclined plane which forms an angle of θ with the horizontal. The coefficient of friction μ depends on the distance x covered by the block as $\mu = ax$, where a is a constant. Find (a) the distance covered by the block till it stops, and (b) it's maximum velocity over the distance traversed.

Sample Questions for Engineering Sciences (PCB, Non-CS Group)

Here are some more sample questions for the topics in Engineering Sciences (PCB, Non-CS Group). To see sample questions for PCA and the rest of the topics of PCB, please refer to the question papers of the last few years, accessible through the admission website:

http://www.isical.ac.in/~admission.

1. A wooden plank of thickness h stands perpendicular to the surface. A bullet of mass m enters the plank perpendicular to it at a height of z above the surface with velocity v_0, as shown in Figure 1. The plank offers a resistive force to the bullet as $F = -kv^2$, where v is the instantaneous velocity of the bullet, and k is a known constant. Assume that the bullet comes out of the plank perpendicular to the plank at the same height z from the surface and subsequently drops to the surface. Let x be the distance of the point where the bullet drops from the base of the plank. Assuming air drag to be zero and acceleration due to gravity as g, find x in terms of m, v_0, k, z, h, and g.

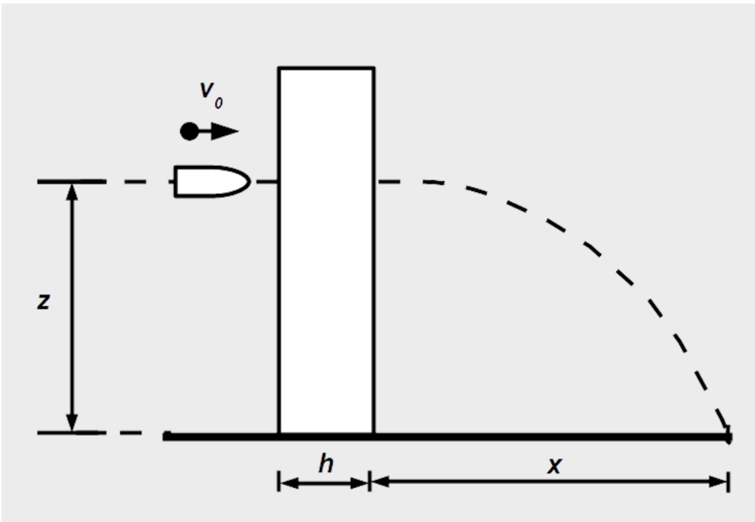


Figure 1: Figure for Question E1.

2. Design a synchronous sequential circuit which detects the occurrence of at least three 1's arriving at the input. The output will go high after it receives three 1's (not necessarily in consecutive clock periods) at the input, and will stay high until it receives three 0's (not necessarily in consecutive clock periods) at the input. When it detects three 0's, the circuit should return to the reset state and start looking again for three 1's. Use J-K flip-flops and other basic logic gates as necessary.

3. A metal rod of length 1 m and uniform cross-section is suspended from ceiling by one of its ends. Assume that the density of the metal is $9000\,\mathrm{kg/m^3}$, Young's modulus is $1.5\times10^{11}\,\mathrm{N/m^2}$, and Posisson's ratio is 0.3. Given that the gravitational acceleration $g = 10\,\mathrm{m/s^2}$, calculate the fractional change in the volume of the rod (i.e. the volume strain) due to its own weight.

4. (a) A Carnot engine absorbs 480 J heat per cycle. The engine operates between a hot source at 320 K and a cold sink at 280 K. What is the work done per cycle output of the engine?

   (b) Assume now that the above engine operates as a refrigerator. Also assume that 700 J heat per cycle is to be removed from the refrigerator. What is the work done per cycle supplied to the refrigerator?

   (c) The inlet and outlet temperatures of a pipe carrying a liquid are 250 K and 260 K respectively. The flow rate of the liquid is 5 kg/s. The pipe is in an ambient temperature of 300 K. The temperature increase at the outlet of the pipe is due to poor insulation and there is no other loss. Calculate irreversible energy associated with the heat leakage due to poor insulation. Assume specific heat of the liquid is 3 kJ/kg K and $\ln(1.04) = 0.04$.

5. A thin and rigid horizontal rod, fixed at both ends, passes through a small hole made in a uniform disc of radius r. The disc remains suspended vertically from the rod and oscillates with small angular amplitude about the rod. Derive expressions for:

   (a) the distance of the hole from the center of the disc for which the time period of such small oscillations is minimum;

   (b) the corresponding minimum time period.

6. A piston can move slowly inside a horizontal cylinder closed at both ends. Initially the piston divides the space inside the cylinder into two equal parts each of volume 10 liters and containing an ideal gas at the same pressure 10 kPa. What amount of work has to be done by an external agent in order to isothermally increase the volume of one part of the gas to 9 times the other part, by slowly moving the piston? Assume that $\ln(5/3) = 0.511$.

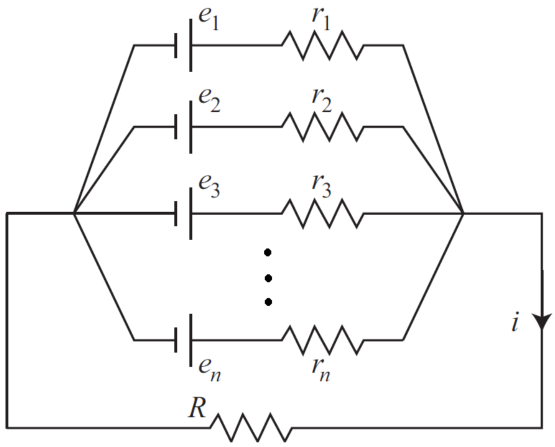
7. Using J-K flip-flops, design a 4-bit Möbius counter which begins with all the output bits as 0's. Starting from the left, these bits toggle to 1, one after another, till all the output bits are 1's. Then these bits become 0, one after another, from the left till all are 0's.

   (a) Provide the state transition table.

   (b) Derive the excitation functions (in the minimized form) of the flip-flops.

   (c) Draw the corresponding circuit.

8. In the figure below, n non-identical cells of emfs $e, 2e, \cdots, ne$ and the respective internal resistances $r, r/2, \cdots, r/n$ are connected in parallel to deliver a current i to an external resistance R. Derive an expression for the current i. What is the value of i when $n >> 1$ and $R = 2n + 1$?



9. (a) A sequential circuit has two input lines x_1 and x_2 and one output line y. The value of y is 1 if the sequence 011 is fed on the x_1 line while x_2 remains 1 for all 3 cycles. Once y is 1, it remains so until x_2 becomes 0. Construct a minimum-row state table for this circuit.

   (b) (i) Draw the Karnaugh-map of an irreducible four-variable Boolean function $f(x_1, x_2, x_3, x_4)$ whose sum-of-products representation consists of the maximum possible number of minterms.

   (ii) Hence prove that no Boolean function with n variables, when expressed in sum-of-products form, requires more than 2^{n-1} product terms.