CS Group (Computer Science)

C1. Consider a standard balance with two pans where weights can only be placed on the left pan, and the object to be weighed on the right pan. Find the minimum number of weights required to weigh any object whose weight in grams could be any integer ranging from 1 to 127. Give precise argument in favor of your answer.

[10]

C2. Let P be a set of n real numbers. For any two real numbers a and b ($a < b$), define

$$R(a, b) = |\{x \in P \mid a \leq x \leq b\}|.$$

(i) Design a suitable data structure $\mathcal{D}$ to store P such that given a, b ($a < b$) as inputs, you can find $R(a, b)$ in time $O(\log n)$. Analyze the time required to construct your data structure $\mathcal{D}$.

(ii) Justify that $O(\log n)$ time is sufficient for reporting $R(a, b)$ using your data structure $\mathcal{D}$.

[5+5]

C3. Let G be a simple undirected graph having n vertices with the property that the average of the degrees of the vertices in G is at least 4. Consider the following procedure:

`Repeatedly delete a vertex from` G `whose degree is at most 2, until there is no such vertex.`

Let H be the resulting graph at the end of this procedure. Then,

(i) Show that the graph H is not empty.

(ii) Using part (i) above, prove that there exist four distinct vertices `a, b, c` and `d` in H such that `deg(a) = deg(b)` and `deg(c) = deg(d)`, where `deg(x)` denotes the degree of the vertex `x`.

[5 + 5]

C4. Let A be a matrix of size $row \times col$. A has to be filled in a spiral clockwise fashion with successive integers from $1, 2, \ldots, row \times col$ starting from the top left corner. For example, a 3×4 matrix should be filled in as follows:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 10 & 11 & 12 & 5 \\ 9 & 8 & 7 & 6 \end{array}$$

Fill in all the blanks in the following code snippet to do the above job. In the answer script, write only the *while* loop with the blanks filled-in.

```
#include <stdio.h>
#include<stdlib.h>
#define RIGHT 0
#define DOWN  1
#define LEFT  2
#define UP    3

void spiralFill ( int **A, int r , int c ) {
   int i, j, top, bottom, left, right, dir, k=1;
   i = j = 0; dir = RIGHT;
   top = 0; bottom = r-1; left = 0; right = c-1;
   while ((top <= _________ ) && (left <= ________)) {
    A[i][j] = k; k++;
    switch (dir) {
    case RIGHT: if(j < ____) ____;
                else{dir = _____; top = ____; i = ____;}
                break;
    case DOWN: if(i < ____) ____;
               else{dir = _____; right = ____; j = ____;}
               break;
    case LEFT: if(j > ____) ____;
               else{dir = _____; bottom = ____; i = ____;}
               break;
    case UP:   if(i > ____) ____;
               else{dir = _____; left = ____; j = ____;}
               break;
```

```
      }
    }
  }

  int main () {
    int **A, row, col, i, j;
    printf("\n Row and column size::>");
    scanf("%d %d",&row,&col);

    A = (int **)calloc(row,sizeof(int *));
    for(i=0;i<row;i++){
      A[i] = (int *)calloc(col,sizeof(int));
    }
   spiralFill(A,row,col);
  }
```

[10]

C5. Given a set S of n integers and a constant k (positive integer), design an algorithm for finding a subset of S of maximum possible size such that the sum of each pair of integers in this subset is not divisible by k. Note that full credit will be given for a polynomial (in n) time algorithm.

[10]

C6. A ternary variable can assume the values 0, 1 or 2, and can be coded with two binary bits as 00, 01 and 10 respectively. A ternary full-adder has three ternary digits X, Y and a carry-in C_{in} as inputs, and produces the ternary sum S (base 3) and the ternary carry-out C_o. For example, if $X = (2)_3$, $Y = (2)_3$ and $C_{in} = (1)_3$, then $S = (2)_3$ and $C_o = (1)_3$. Design a circuit for this ternary full adder using binary gates as well as binary half and full adders.

[10]

C7. (a) Consider the single precision (i.e., 32-bit) floating point representation of numbers in the normalized form where 8 bits are used for the exponent with the bias of 127.

(i) What is the binary representation of -10.4 in the above form? The steps followed to arrive at the representation must be shown.

(ii) How many different floating point numbers can be represented in the above form, lying strictly between 2^{-18} and 2^{-17} (i.e., excluding 2^{-18} and 2^{-17})?

[3+2]

(b) Consider a 4-way set associative cache mapping, in which the cache blocks are grouped into sets and each set has 4 blocks. There are 16 cache blocks in total. The following memory block requests arrive in order when the cache memory is empty:

8, 4, 8, 3, 0, 7, 12, 64, 216, 8.

If a set is full, the Least Recently Used (LRU) policy is used to replace a block in that set to make room for the present request.

(i) Show the cache configuration (along with intermediate configurations) on meeting the above memory requirements.

(ii) What is the hit ratio?

[4+1]

C8. Consider the following language L over the alphabet $\Sigma = \{a, b, c\}$.

$$L = \{a^i b^j c^k \mid i, j, k \geq 0, \text{ and if } i = 1 \text{ then } j = k\}$$

(i) Show that L is not regular.

(ii) Show that L is a context-free language.

[7+3]

C9. In relational algebra, for any pair of relations R_1 and R_2, the `standard division` operation is denoted by $\div$ and defined as follows:

$$R_1 \div R_2 = \pi_{A^*}(R_1) \;-\; \pi_{A^*}((\pi_{A^*}(R_1) \times R_2) \;-\; R_1)$$

Here, A^* denotes the attributes that are in R_1 but not in R_2. The following is an example of a `standard division` operation.

R_1

A	**B**	**C**
1	Red	Leaf
2	Green	Leaf
2	Blue	Leaf
3	Red	Leaf
3	Red	Stem
4	Pink	Leaf
5	Black	Stem

R_2

C
Leaf
Stem

$R_1 \div R_2$

A	**B**
3	Red

(i) Cite a working example wherein a pair of relations R_1 and R_2 satisfies the following equation:

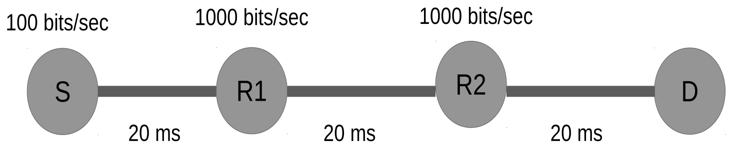
$$(R_1 \div R_2) \times R_2 = R_1.$$

(ii) Recall that the notations $\bowtie$, $⟕$ and $⟖$ denote **`natural join, left outer join`**, and **`right outer join`** operations, respectively. For any arbitrary pair of relations R_1 and R_2, prove that if $(R_1 \div R_2) \times R_2 = R_1$ holds, then the following will also hold.

$$(R_1 ⟕ R_2) - (R_1 \bowtie R_2) \equiv (R_1 ⟖ R_2) - (R_1 \bowtie R_2).$$

[4+6]

C10. Consider sending a message of 10,000 bits from the source node S to the destination node D passing through the two routers $R1$ and $R2$ as shown in the figure. Each of the three links on the path has a propagation delay of 20 ms. Node S has a transmission rate of 100 bits/sec, and each of $R1$ and $R2$ has a transmission rate of 1000 bits/sec. Assume that: (i) each router uses store-and-forward packet switching; (ii) the size of the header is negligible compared to the packet size; (iii) except transmission and propagation delays, all other delay components are negligible.



(i) Find the end-to-end latency of the message when it is sent as a whole.

(ii) Find the end-to-end latency of the message when it is broken into 10 packets each of size 1000 bits, and then transmitted to the destination.

[4+6]

Non-CS Group (Mathematics)

NC1. Consider a standard balance with two pans where weights can only be placed on the left pan, and the object to be weighed on the right pan. Find the minimum number of weights required to weigh any object whose weight in grams could be any integer ranging from 1 to 127. Give precise argument in favor of your answer.

[10]

NC2. Consider three real numbers $a \geq b \geq c > 0$. If $(a^x - b^x - c^x)(x-2) > 0$ for any rational number $x \neq 2$, show that

(i) a, b and c can be the lengths of the three sides of a triangle ABC;

(ii) ABC is a right-angled triangle.

[3+7]

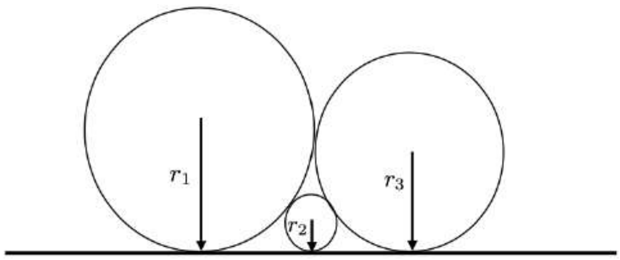
NC3. Consider a two-player game between Alice and Bob, in which the players take turns to roll a fair six-faced die. Alice rolls the die first. Then Bob rolls the die and he wins if he gets the same outcome as Alice. Otherwise, Alice rolls the die again and she wins if she gets the same outcome as Bob. The game continues in this way, and it terminates as soon as one player gets the same outcome as obtained by the opponent in the previous roll of the die. The player who succeeds in doing so first is the winner.

(i) Find the probability that the game does not terminate after the first three rolls (two by Alice and one by Bob) of the die.

(ii) What is the probability that Alice will win the game?

[3+7]

NC4. In the figure below, there are three circles touching each other externally and also touching the line below. Let r_1, r_2 and r_3 be the radii of the three circles as shown in the figure. If $r_1 = 25$ and $r_3 = 9$, then find r_2.



[10]

NC5. Let G be a group generated by a and b such that $\operatorname{ord}(a) = n$, $\operatorname{ord}(b) = 2$ and $ab = ba^{-1}$, where n is a positive integer, $b \notin \langle a \rangle$ and $\operatorname{ord}(x)$ denotes the order of the element x.

(i) Prove that for any positive integer k, $ab^k = ba^{-k}$.

(ii) Let H be a cyclic subgroup of $\langle a \rangle$. Show that H is a normal subgroup of G.

[5+5]

NC6. Let p be an odd prime and let $n = (p-1)(p+1)$.

(i) Show that p divides $n2^n + 1$.

(ii) Show that there are infinitely many integers m such that p divides $m2^m + 1$.

[6+4]

NC7. Let G be a cubic graph, that is, every vertex has degree exactly 3.

(i) Prove that the number of vertices of G cannot be 101.

(ii) Prove that if G contains 100 vertices, then it contains a bipartite subgraph that has at least 75 edges.

[3+7]

NC8. (i) Calculate the number of different ways you can divide $2n$ elements of the set $S = \{1, 2, \ldots, 2n\}$ to form n disjoint subsets, each containing a pair of elements.

(ii) Calculate the number of different ways in which the above division can be done if each subset is required to contain an even number and an odd number.

[6+4]

NC9. Consider a 4×4 positive semi-definite matrix A with all diagonal elements equal to 1 and all off-diagonal elements equal to ρ.

(i) If $\rho < 0$, show that the largest eigenvalue of A cannot exceed $4/3$.

(ii) Give an eigenvector of A other than $(1, 1, 1, 1)^\top$.

[7+3]

NC10. Let $a > 0$ and $x_1 > 0$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for all $n \in \mathbb{N}$. Show that

(i) $x_n > \sqrt{a}$ for all $n \geq 2$;

(ii) the sequence $\{x_n : \; n \geq 1\}$ converges to $\sqrt{a}$.

[3+7]