The following notations are used throughout the question paper.

- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{N}$ denotes the set of natural numbers.
- $\mathbb{C}$ denotes the set of complex numbers.
- For a real number x, $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.
- For a real number x, $\lceil x \rceil$ denotes the least integer $\geq x$.

1. The roots of the equation $x^4 + x^3 + x^2 + x + 1 = 0$ are

   (A) the vertices of a square.
   (B) (some of) the vertices of a regular pentagon.
   (C) (some of) the vertices of a regular hexagon.
   (D) (some of) the vertices of a regular heptagon.

2. Let $C = \{\binom{1}{2}, \binom{2}{1}\}$ be a basis of $\mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\binom{x}{y} = \binom{x+y}{x-2y}$. If $T(C)$ represents the matrix of T with respect to the basis C, then which of the following is true?

   (A) $T(C) = \begin{pmatrix} -3 & -2 \\ 3 & 1 \end{pmatrix}$ (B) $T(C) = \begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix}$
   (C) $T(C) = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$ (D) $T(C) = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}$

3. Which of the following pairs of regular expressions are not equivalent?

   (A) $xx^\star$ and $x^\star x$ (B) $x(yx)^\star$ and $(xy)^\star x$
   (C) $x(xx)^\star$ and $(xx)^\star x$ (D) $(xy)^\star$ and $x^\star y^\star$.

4. Consider the disc on the (x, y) plane with centre at $(0, 1)$ and radius 1 unit. Let A be the highest point on the disk, that is, $(0, 2)$. If this disk rolls on the x-axis such that its centre is now at $(1, 1)$, then the new coordinates of A are

(A) $\big(1 + \sin(1/2\pi), 1 + \cos(1/2\pi)\big)$
(B) $\big(1 + \sin(1), 1 + \cos(1)\big)$
(C) $\big(1 + \sin(1/2\pi), 1 - \cos(1/2\pi)\big)$
(D) $(1, 2)$

5. Let $f : [0, 1] \to (0, \infty)$ be a continuous function. Define $g(x) = \int_0^x f(t)\,dt$ for $0 \le x \le 1$. If $g(1) = 1$, the equation $g(x) = 1/2$

(A) may not have a solution.
(B) may have more than one solution.
(C) will have at least two solutions.
(D) will have a unique solution.

6. Let $A = \langle a_1 a_0 \rangle$ and $B = \langle b_1 b_0 \rangle$ be two 2-bit numbers and $C = \langle c_3 c_2 c_1 c_0 \rangle$ be a 4-bit number, where $c_0 = a_0$, $c_1 = a_1 \oplus b_0$, $c_2 = b_1 \oplus a_1 b_0$, and $c_3 = a_1 b_0 b_1$ ('$\oplus$' denotes the XOR operation). If $X + Y$ and XY denote the arithmetic sum and product (of X and Y) respectively, which of the following is true?

(A) $C = A + B$
(B) $C = AB$
(C) $C = 2A + B$
(D) $C = A + 2B$

7. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of n natural numbers and let n be odd. For any finite set $X \subseteq \mathbb{N}$, $\mathsf{Max}(X)$ and $\mathsf{Min}(X)$ denote the maximum and minimum elements in X respectively. Let $|X|$ denote the number of elements in X. If $|X|$ is odd, $\mathsf{Median}(X)$ returns the $(|X|+1)/2$-th smallest element in X. Consider the following pseudo-code, which takes as input the set A and outputs a set S.

```
function FUN(A)
    Y := {a_1, a_2, a_3}
    m := Median(Y)
    Y := Y \ {m}
    for i = 4 to n do
        Y := Y ∪ {a_i}
        m := Median(Y)
        Y := Y \ {m}
    S := Y
    Return S
```

Which of the following is not correct?

(A) $\mathsf{Median}(A) = \mathsf{Median}(A \setminus S)$

(B) $\mathsf{Max}(A) = \mathsf{Max}(S)$

(C) $\mathsf{Min}(A) = \mathsf{Min}(S)$

(D) There is an $x \in S$, such that $\mathsf{Min}(A) < x < \mathsf{Max}(A)$.

8. The $\lim_{n\to\infty} \dfrac{1 + \sqrt{2} + \sqrt[3]{3} + \cdots + \sqrt[n]{n}}{n}$

(A) is equal to 0.

(B) is equal to 2.

(C) is equal to 1.

(D) does not exist.

9. Let A and B be two arrays of size n, each containing the numbers $\{1, 2, \ldots, n\}$ in some order. The operations allowed on the entries of the arrays are comparisons between two entries and swapping two entries. Consider the best possible algorithm to transform the array A to array B using the above two operations only. The total number of operations in which the above algorithm can perform this task in the worst case is

(A) a constant (B) n
(C) $n!$ (D) none of these

10. For $w \in \mathbb{C}$, let $Re(w)$ and $Im(w)$ denote the real and imaginary parts of w, respectively. Suppose $z \in \mathbb{C}$ is such that $|z| = 1$. Which of the following is necessarily true?

(A) $Re(z)$ is rational implies that $Re(z^2)$ is rational.
(B) $Re(z^2)$ is rational implies that $Re(z)$ is rational.
(C) $Im(z)$ is rational implies that $Im(z^2)$ is rational.
(D) $Im(z^2)$ is rational implies that $Im(z)$ is rational.

11. Suppose A is an $n \times n$ matrix for some $n \geq 2$. Then $A^n = 0$ if and only if

(A) $A = 0$ (B) all the eigenvalues of A are zero.
(C) A is singular. (D) Rank$(A) \leq 1$

12. Let A_1 and A_2 be two arbitrarily chosen subsets of $\{1, 2, \ldots, n\}$. The number of ways such that $A_1 \cap A_2 = \emptyset$ is

(A) 2^n (B) $4^n - 3^n$ (C) 3^n (D) $3^n - 1$

13. The following program Check takes as input a graph and an integer and outputs True or False.

```
function Check(G = (V, E), k)
    if for all vertices u and v in V, edge (u, v) ∈ E then
        if the number of vertices is at least k then
            Return True
        else
            Return False
    Let u and v be vertices in V such that (u, v) ∉ E
    A := V \ {v}
    Return Check(A, k)
```

Which of the following is true about the program?

(A) There exists an input graph G and an integer k such that the program Check(A, k) never halts.

(B) For all input graphs G and integers k, the program Check(A, k) outputs True if and only if there are at least k vertices in the graph G.

(C) For all input graphs G and integers k, the program Check(A, k) outputs True if and only if there is a subset S of k vertices in the graph and all the vertices in S are connected to each other by an edge.

(D) For all input graphs G and integers k, the program Check(A, k) outputs True if and only if for all subsets S of k vertices in the graph, all the vertices in S are connected to each other by an edge.

14. The maximum value of the function $f(x) = \frac{x}{2} - \frac{1}{1+x^2}$ on the set $\{x \in \mathbb{R} : x(x-1) \leq 12\}$ is

(A) 0 (B) 33/17 (C) $-8/5$ (D) -1

15. If P and Q are two propositions, then the expression

$$(P \wedge (P \to Q)) \to Q \text{ is}$$

(A) always true. (B) true for only one assignment.
(C) always false. (D) false for only one assignment.

16. Let $n \geq 2$. Suppose $\alpha_1, \alpha_2, \cdots, \alpha_n$ are real numbers such that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Let

$$P(x) = (x - \alpha_1)^2(x - \alpha_2)\cdots(x - \alpha_n),\ x \in \mathbb{R}.$$

If P' is the derivative of P, then $P'(x)$ has

(A) no real root.
(B) n real roots, the smallest of which is α_1.
(C) $n-1$ real roots, the smallest of which is α_1.
(D) n real roots, the smallest of which is strictly larger than α_1.

17. Let $p(x)$ be a polynomial in x of degree $k \geq 0$. For $n = 1, 2, \ldots,$ define $a_n = p(n+1) - p(n)$. If $\{a_n\}_{n=1}^{\infty}$ is in AP, then we must have

(A) $k \leq 1$ (B) $k \leq 2$ (C) $k \geq 3$ (D) $k \geq 4$

18. Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$. Then

(A) $S_{2^n} \geq \frac{n}{2}$ for every $n \geq 1$
(B) S_n is a bounded sequence.
(C) $|S_{2^n} - S_{2^{n-1}}| \to 0$ as $n \to \infty$
(D) $\frac{S_n}{n} \to 1$ as $n \to \infty$

19. If

$$2021^{2024} = 100\, m + n\,,$$

for some $m \in \{0, 1, 2, \ldots\}$ and $n \in \{0, 1, \ldots, 99\}$, then the value of n is

(A) 21 (B) 31 (C) 61 (D) 81

20. Let G be a simple undirected graph. For a vertex $u \in V(G)$, let $N_G(u)$ denote the set of neighbours of u in G, i.e., $N_G(u) = \{v : uv \in E(G)\}$. Let H be the undirected graph having $V(H) = V(G)$ and $E(H) = \{uv : N_G(u) \cap N_G(v) \neq \emptyset\}$. If G is a connected bipartite graph, then the number of connected components of H:

(A) is exactly one
(B) can be either one or two
(C) is exactly two
(D) can be any positive integer

21. What is the maximum number of nonzero values in the adjacency matrix of a directed graph that has 36 vertices, but no self-loops?

(A) 36 (B) 180 (C) 630 (D) 1260

22. Let A and B be two real symmetric square matrices with the property that both A and B have the same set of eigenvectors. Then, which of the following is always true?

(A) $A = B$
(B) $AB = BA$
(C) $A = B^{-1}$
(D) $AA^T = BB^T$

23. Consider the following pseudo-code.

---

function $f(n, m)$
    if $m = 0$ **then**
        return 1
    if $m = 1$ **then**
        return n
    Return $f(n, \lfloor \frac{m}{2} \rfloor) \cdot f(n - \lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil)$

Which of the following is true?

(A) $f(n, m) = n^m$
(B) $f(n, m) = (n + 1 - m)^m$
(C) $f(n, m) = \frac{n!}{(n-m)!}$
(D) $f(n, m) = \left(\frac{n(n-1)}{n-m}\right)^m$

24. The value of

$$\int_0^1 \int_{x^2}^1 e^{y^{3/2}} \, dy \, dx$$

is

(A) $\frac{2}{3}(e - 1)$
(B) $e - 1$
(C) $\frac{2e}{3}$
(D) $\frac{2e}{3} - \frac{3}{2}$

25. Consider the curves

$$\begin{aligned} y_1(x) &= \frac{1}{2}x + \frac{1}{2}x^5, \\ y_2(x) &= \frac{1}{2}x^3 + \frac{1}{2}x^7, \\ y_3(x) &= 1 + \frac{1}{2}x^3 + \frac{1}{2}x^7. \end{aligned}$$

Which of the following is true?

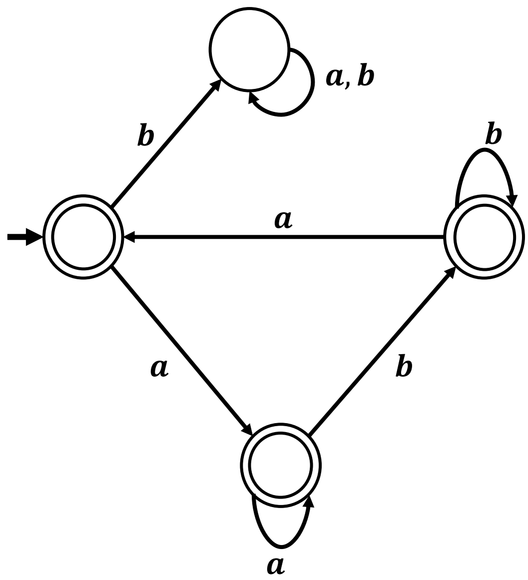
(A) y_1 and y_2 meet at 2 points, y_1 and y_3 meet at 1 point.

(B) y_1 and y_2 meet at 3 points, y_1 and y_3 meet at 2 points.

(C) y_1 and y_2 meet at more than 3 points, y_1 and y_3 meet at 1 point.

(D) y_1 and y_2 meet at 3 points, y_1 and y_3 meet at 1 point.

26. Which of the following regular expressions is equivalent to the non-deterministic finite automaton shown in the figure below?



(A) $(a^*b^*a)^*(\epsilon \cup aa^*b^*)$

(B) $(aa^*bb^*a)(\epsilon \cup aa^* \cup bb^*)$

(C) $(aa^*bb^*a)(\epsilon \cup aa^* \cup aa^*bb^*)$

(D) $(aa^*bb^*a)^*(\epsilon \cup aa^*b^*)$

27. The 26 letters of the alphabet are randomly permuted. What is the probability that at least two of the three letters I, N, D come next to each other?

(A) $\dfrac{72}{325}$ (B) $\dfrac{3}{26}$ (C) $\dfrac{3}{13}$ (D) $\dfrac{3}{25 \times 26}$

28. There are two coins in a box. One of the coins is a fair coin and for the other, the probability of head is $\frac{2}{3}$. A coin is chosen at random from the box and tossed 5 times. Given that at least 4 heads were obtained, what is the conditional probability that the fair coin was chosen?

(A) $\frac{729}{2521}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{243}{275}$

29. For $x, y \geq 0$, consider the regions $A = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x^4 + y^4 > 1\}$. Then

(A) $A \subset B$ (B) $B \subset A$
(C) $A \cap B = \emptyset$ (D) $A \cup B = \mathbb{R}^2$

30. Let $S = \{\left(\begin{smallmatrix} a & b\sqrt{5} \\ -b\sqrt{5} & a \end{smallmatrix}\right) : a, b \in \mathbb{Z}\}$. The number of $A \in S$ such that A^{-1} is also in S is

(A) 2 (B) 4
(C) more than 4 but finite (D) infinite