1. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with derivative f'. Suppose $f(0) = 0$ and $f'(x) > f(x)$ for all $x \in \mathbb{R}$.

   (a) Prove that $e^{-x} f(x)$ is an increasing function of x on $(0, \infty)$.

   (b) Show that $\lim_{x \to \infty} f(x) = \infty$.

2. Find the number of functions $f : \{1, 2, \ldots, 2n\} \to \{1, 2, \ldots, 2n\}$ satisfying

$$f(i) \neq i \text{ and } f(f(i)) = i \text{ for all } i = 1, 2, \ldots, 2n\,.$$

3. Suppose A is an $n \times n$ real matrix which has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$ corresponding to real positive eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

   (a) Let V be the $n \times n$ matrix whose i-th column is v_i for all $i = 1, 2, \ldots, n$, and D be the diagonal matrix whose i-th diagonal entry is λ_i for $i = 1, 2, \ldots, n$. Show that

$$V^{-1}AV = D.$$

   (b) Show that there exist 2^n distinct matrices $B_1, B_2, \ldots, B_{2^n}$ such that $B_j^2 = A$ for all $j = 1, 2, \ldots, 2^n$.

4. Let $\rho \in (-1, 1)$, and $X_0, X_1, X_2, \ldots$ be a sequence of random variables satisfying

$$X_i = \rho X_{i-1} + \varepsilon_i \text{ for all } i \geq 1.$$

Suppose $\varepsilon_1, \varepsilon_2, \ldots$ have a common mean 0 and a common variance τ^2, and X_0 has mean 0 and variance $\sigma^2 = \dfrac{\tau^2}{1-\rho^2}$. Assume further that $X_0, \varepsilon_1, \varepsilon_2, \ldots$ are independent.

(a) Find $\mathrm{Var}(X_i)$ for $i \geq 1$.

(b) Find $\mathrm{Cov}(X_i, X_j)$ for $j > i \geq 0$, and show that it is a function of $j - i$.

5. Suppose $X_1, X_2, \ldots, X_{2n}$ are independent Exponential random variables with mean $1/\lambda$, where $\lambda > 0$. Let $X_{(1)} \leq \cdots \leq X_{(2n)}$ denote the order statistics of $X_1, \ldots, X_{2n}$.

(a) Show that for all $x > 0$,

$$P\left(X_{(n)} \leq x\right) = P\left(\left[\sum_{i=1}^{2n} \mathbf{1}\{X_i \leq x\}\right] \geq n\right).$$

Here $\mathbf{1}\{A\}$ denotes the indicator function of an event A, which takes the value 1 if A occurs, and 0 otherwise.

(b) Hence or otherwise, prove that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(X_{(n)} \leq \frac{1}{\lambda}\log_e 2 - \varepsilon \right) = 0\,.$$

6. Suppose π is a permutation of $\{1, \ldots, n\}$ where $n \geq 2$. A pair (i, j) with $i < j$ is called an *inversion with respect to* π if $\pi_i > \pi_j$.

   Let X denote the number of inversions with respect to a permutation chosen at random from all possible permutations of $\{1, \ldots, n\}$. Find $\mathrm{E}(X)$.

7. A population consists of units $\{1, 2, \ldots, n+1\}$ for some $n \geq 1$. Consider the following sampling procedure.

   - Start with $\mathcal{S}_0 = \{1, 2, \ldots, n\}$, consisting of the first n units.
   - Generate U from Uniform $(0, 1)$.
     - If $U > \dfrac{n}{n+1}$, define $\mathcal{S}_1 = \mathcal{S}_0$.
     - Otherwise, remove one of the units uniformly at random from $\mathcal{S}_0$ (independently of U), and add unit $(n+1)$ to it to create $\mathcal{S}_1$.

   Determine, with justification, whether $\mathcal{S}_1$ is a simple random sample of size n from the population.

8. Consider the following bivariate data.

i	1	2	3	4	5	6	7	8	9	10	11	12
x_i	21	21	21	21	21	35	35	35	35	35	35	35
y_i	33	93	34	48	19	58	14	32	12	44	45	62

   Determine, with justification, the values of a and b that minimise the sum of absolute errors

$$\sum_{i=1}^{12} \left| y_i - a - bx_i \right|.$$

9. Let $X_1, X_2, \ldots, X_n$ be independent random variables having common probability density function f_θ, $\theta \in \{0, 1\}$, where

$$f_0(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_1(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose we wish to test $H_0 : \theta = 0$ vs. $H_1 : \theta = 1$ at significance level α, where $0 < \alpha < 1$. Find the rejection region of the most powerful test based on $X_1, X_2, \ldots, X_n$ in terms of quantiles of standard distributions.

10. Let θ_1, θ_2, and θ_3 denote the three angles of a triangle, measured in degrees (i.e., $\theta_i > 0$ for $i = 1, 2, 3$, and $\theta_1 + \theta_2 + \theta_3 = 180$). Suppose that each angle is measured by an instrument that is subject to independent additive $N(0, \sigma^2)$ error, and the measured values of θ_1, θ_2, and θ_3 are found to be $y_1 = 83$, $y_2 = 47$, and $y_3 = 56$, respectively. Find the maximum likelihood estimates of θ_1, θ_2, and θ_3.