GROUP A

1. Let a and b be real numbers. Show that there exists a unique 2×2 real symmetric matrix A with $\text{trace}(A) = a$ and $\det(A) = b$ if and only if $a^2 = 4b$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function and suppose that for some $n \geq 1$,

$$f(1) = f(0) = f^{(1)}(0) = f^{(2)}(0) = \cdots = f^{(n)}(0) = 0,$$

where $f^{(k)}$ denotes the k-th derivative of f for $k \geq 1$. Prove that there exists $x \in (0,1)$ such that $f^{(n+1)}(x) = 0$.

3. Consider an urn containing 5 red, 5 black, and 10 white balls. If balls are drawn *without* replacement from the urn, calculate the probability that in the first 7 draws, at least one ball of each colour is drawn.

GROUP B

4. Let $X_1, X_2, \ldots, X_n$ be independent random variables, with X_i having probability mass function

$$P(X_i = k) = \left(\frac{i}{i+1}\right)^k \frac{1}{i+1}, \text{ for } k = 0, 1, 2, \ldots$$

and for all $i = 1, \ldots, n$. Let $M = \min\{X_i : 1 \leq i \leq n\}$. Derive the probability mass function of M.

5. The lifetime in hours of each bulb manufactured by a particular company follows an independent exponential distribution with mean λ. To test the null hypothesis $H_0 : \lambda = 1000$ against the alternative $H_1 : \lambda = 500$, a statistician sets up an experiment with 50 bulbs, with 5 bulbs in each of 10 different locations, to examine their lifetimes.

   To get quick preliminary results, the statistician decides to stop the experiment as soon as one bulb fails at each location. Let Y_i denote the lifetime of the first bulb to fail at location i. Obtain the most powerful test of H_0 against H_1 based on $Y_1, Y_2, \ldots, Y_{10}$, and compute its power.

6. Suppose you have a 4-digit combination lock, but you have forgotten the correct combination. Consider the following three strategies to find the correct one:

   (i) Try the combinations consecutively from 0000 to 9999.

   (ii) Try combinations using simple random sampling *with* replacement from the set of all possible combinations.

   (iii) Try combinations using simple random sampling *without* replacement from the set of all possible combinations.

   Assume that the true combination was chosen uniformly at random from all possible combinations. Determine the expected number of attempts needed to find the correct combination in all three cases.

7. Consider independent observations $\{(y_i, x_{1i}, x_{2i}) : 1 \le i \le n\}$ from the regression model

$$y_i = \beta_1 x_{1i} + \beta_2 x_{2i} + \epsilon_i,\ i = 1, \ldots, n\,,$$

   where x_{1i} and x_{2i} are scalar covariates, β_1 and β_2 are unknown scalar coefficients, and ϵ_i are uncorrelated errors with mean 0 and variance $\sigma^2 > 0$. Instead of using the correct model, we obtain an estimate $\hat{\beta}_1$ of β_1 by minimizing

$$\sum_{i=1}^{n} (y_i - \beta_1 x_{1i})^2.$$

   Find the bias and mean squared error of $\hat{\beta}_1$.

8. Let $\theta > 0$ be an unknown parameter, and $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with density

$$f(x) = \begin{cases} 2x/\theta^2 & , 0 \le x \le \theta\,, \\ 0 & , \text{otherwise.} \end{cases}$$

   Find the maximum likelihood estimator of θ and its mean squared error.