Notations and Abbreviations

The following are used throughout the question paper.

$\mathbb{R}$	Set of real numbers
$\det(A)$	Determinant of the matrix A
$\log x$	Natural logarithm of x
$\exp(x)$	e^x
E	Expectation
Var	Variance
Cov	Covariance
c.d.f.	cumulative distribution function
Φ	c.d.f. of the standard Normal distribution
i.i.d.	independent and identically distributed

1. The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = e^x - e^{2x} - 6e^{3x}$$

   (A) has no real root.

   (B) has exactly three real roots.

   (C) has exactly one real root.

   (D) has exactly two real roots.

2. Let S be the set of all ordered pairs (x, y) such that x, y are integers satisfying the relation

$$x^2 - y^2 = 154.$$

   What is the number of elements in S?

   (A) 0 (B) 4 (C) 8 (D) 2

3. Suppose A and B are two 5×5 matrices with real entries. Consider the following two statements:

   I. $\det(A + B) \geq \det(A) + \det(B)$.

   II. $\mathrm{trace}(AB) \leq \mathrm{trace}(A)\,\mathrm{trace}(B)$.

   Then

   (A) both I and II are false.

   (B) both I and II are true.

   (C) II is true but I is false.

   (D) I is true but II is false.

4. Suppose $(a_n)_{n\geq 1}$ is a convergent sequence of real numbers. Let

$$b_n = a_n + \frac{1}{n}, \quad c_n = a_n + \frac{2n}{n+1} \quad \text{for all } n \geq 1.$$

Define

$$(x_n)_{n\geq 1} = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots),$$
$$(y_n)_{n\geq 1} = (a_1, c_1, a_2, c_2, \ldots, a_n, c_n, \ldots).$$

Then

(A) neither $(x_n)_{n\geq 1}$ nor $(y_n)_{n\geq 1}$ is convergent.

(B) both $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are convergent.

(C) $(x_n)_{n\geq 1}$ is convergent but $(y_n)_{n\geq 1}$ is not.

(D) $(y_n)_{n\geq 1}$ is convergent but $(x_n)_{n\geq 1}$ is not.

5. Identify the smallest interval that contains all the elements of

$$A = \left\{ \frac{m}{n} + \frac{4n}{m} : m, n \text{ are positive integers} \right\}.$$

(A) $[5, \infty)$ (B) $(4, \infty)$ (C) $[4, \infty)$ (D) $(0, \infty)$

6. Let $f(x) = \sqrt{x} \log x$ for $x \in (0, \infty)$. Which of the following statements is **not** true?

(A) f has a local maxima.

(B) f is differentiable everywhere on $(0, \infty)$.

(C) $f(x)$ has a limit as $x \to 0+$.

(D) f has a local minima.

7. How many solutions does the following system of equations have?

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 2x_3 &= 1 \\ x_1 - 3x_2 + 4x_3 &= 2 \end{aligned}$$

(A) 2 (B) 1 (C) 0 (D) 3 or more

8. Let $\alpha > 0$. Find the value of

$$\lim_{n\to\infty} \frac{\Gamma(2\alpha/n)}{\left(\Gamma(\alpha/n)\right)^2} \int_0^1 x^{\frac{\alpha}{n}+1}(1-x)^{\frac{\alpha}{n}-1} \mathrm{d}x\,,$$

where $\Gamma(x)$ denotes the Gamma function evaluated at x.

(A) $\alpha/2$ (B) $1/2$ (C) 1 (D) 0

9. How many permutations of the twenty-six letters $A, B, \ldots, Z$ of the alphabet are there in which Y is adjacent to at least one of the five vowels A, E, I, O, U?

(A) $5550 \cdot (23)!$ (B) $230 \cdot (24)!$

(C) $5 \cdot (25)!$ (D) $10 \cdot (25)!$

10. If A is a 3×3 real symmetric matrix with trace$(A) = 0$, then what is the set of possible values of rank(A)?

(A) $\{0, 1, 3\}$ (B) $\{0, 2, 3\}$

(C) $\{0, 1, 2\}$ (D) $\{0, 1, 2, 3\}$

11. Let A and B be two events such that

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{4}{9}, \text{ and } P(A \mid B) = \frac{3}{4}.$$

Let X be the random variable which takes the value 1 if A occurs and 0 otherwise. Similarly, Y is the random variable which takes the value 1 if B occurs and 0 otherwise. Consider the following statements:

I. $\mathrm{E}(XY) = \frac{1}{3}$.

II. $P\big(XY = X^2Y^2\big) = 1$.

Then

(A) both I and II are wrong. (B) both I and II are correct.

(C) I is correct and II is wrong. (D) I is wrong and II is correct.

12. Suppose (X_1, X_2) follows a Bivariate Normal distribution with

$$\mathrm{E}(X_1) = \mathrm{E}(X_2) = 0, \mathrm{Var}(X_1) = \mathrm{Var}(X_2) = 1, \mathrm{Cov}(X_1, X_2) = \rho$$

for some $0 \leq \rho \leq 1$. Then $\mathrm{E}(X_1^2 X_2^2)$ equals

(A) 2ρ. (B) $1 + 2\rho^2$. (C) ρ^2. (D) $\rho + 2\rho^2$.

13. A random variable U with c.d.f. F *stochastically dominates* a random variable V with c.d.f. G if $F(t) \leq G(t)$ for every $t \in \mathbb{R}$. Suppose

$$X \sim N(1,1), \quad Y \sim N(2,1), \quad Z \sim N(2,2).$$

Consider the following statements:

I. Y stochastically dominates X.

II. Z stochastically dominates Y.

Then

(A) both I and II are false. (B) I is true but II is false.

(C) both I and II are true. (D) II is true but I is false.

14. Let $X_1, \ldots, X_7$ be i.i.d. with c.d.f. F and density f. Then the density of the sample median is

(A) $140F(x)^3(1-F(x))^3$. (B) $7F(x)^3(1-F(x))^3 f(x)$.

(C) $140F(x)^3(1-F(x))^3 f(x)$. (D) $840F(x)^6 f(x)$.

15. Suppose two players A and B take turns in conducting a sequence of trials. Each trial consists of rolling a pair of fair dice, and recording the sum of the two numbers obtained. The first player who obtains exactly 7 in a trial is the winner. If A rolls first, what is the probability that B will win?

(A) $\frac{1}{2}$ (B) $\frac{6}{11}$ (C) $\frac{5}{11}$ (D) $\frac{7}{36}$

16. Suppose a random sample of four observations is drawn from the Poisson distribution with mean $\lambda > 0$. Let $\overline{X}$ denote the sample mean. What is $P\left(\overline{X} < \frac{1}{2}\right)$?

(A) $(1 + 4\lambda + 8\lambda^2)e^{-4\lambda}$ (B) $(4\lambda + 1)e^{-4\lambda}$

(C) $(2\lambda + 1)e^{-2\lambda}$ (D) $e^{-4\lambda}$

17. Graduating students at a certain college bring zero, one, or two parents to attend the graduation ceremony with equal probability, independently of other students. In a class of 600 graduating students, let N be the number of parents who attend. What is the variance of N?

(A) 200 (B) 800 (C) 600 (D) 400

18. If $A_1, A_2, A_3, \ldots$ are independent events with

$$P(A_n) = p_n \text{ for all } n \geq 1,$$

then what does $P\left(\bigcup_{n=1}^{\infty} A_n\right)$ equal?

(A) $1 - \prod_{n=1}^{\infty} p_n$ (B) $\prod_{n=1}^{\infty}(1 - p_n)$

(C) $\sum_{n=1}^{\infty} p_n$ (D) $1 - \prod_{n=1}^{\infty}(1 - p_n)$

19. Suppose $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \text{Normal}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$. Find the value of

$$P(X_1 + 3X_2 > 2\sqrt{13}).$$

(A) $1 - \Phi(2\sqrt{13})$

(B) $1 - \Phi(\sqrt{13/7})$

(C) $1 - \Phi(1)$

(D) $1 - \Phi(-1)$

20. Suppose X_1 and X_2 are independent Bernoulli(1/2) random variables. If $M = \max\{X_1, X_2\}$, find the value of $\text{Cov}(X_2, M)$.

(A) $\frac{1}{8}$

(B) $\frac{1}{2}$

(C) $-\frac{1}{4}$

(D) 0

21. Suppose $X_1, X_2, \ldots, X_n$ are independent Uniform$(0, 1)$ random variables. Find the smallest value of n such that

$$P\big(\max\{X_1, X_2, \ldots, X_n\} \geq 0.99\big) \geq 0.95.$$

Here $\lceil a \rceil$ denotes the smallest integer larger than or equal to a.

(A) $\left\lceil \frac{\log(0.05)}{\log(0.99)} \right\rceil$

(B) $\left\lceil \frac{\exp(0.99)}{\exp(0.05)} \right\rceil$

(C) $\left\lceil \frac{0.99}{0.05} \right\rceil$

(D) $\left\lceil \frac{0.99}{1 - 0.05} + \frac{0.95}{1 - 0.01} \right\rceil$

22. Each worker in a factory takes lunch in a canteen with probability 0.1 and elsewhere with probability 0.9, independently of other workers. If n workers work in the factory, the factory management wishes to find α_n such that

$$\lim_{n\to\infty} P\left(\text{At most } \alpha_n \text{ workers take lunch in the canteen}\right) = 0.95.$$

Which of the following values of α_n ensures the above?

(A) $\alpha_n = n \times 0.1$

(B) $\alpha_n = n \times 0.1 + \sqrt{n} \times 0.3 \times \Phi^{-1}(0.95)$

(C) $\alpha_n = n \times 0.1 + \sqrt{n} \times 0.3 \times \Phi^{-1}(0.975)$

(D) $\alpha_n = n \times 0.1 + \sqrt{n} \times 0.09 \times \Phi^{-1}(0.95)$

23. Suppose $X \sim \text{Uniform}(0, \theta)$, $\theta > 0$, and $Y = -\frac{1}{\theta} \log\left(\frac{X}{\theta}\right)$. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution of X. Let $M = \max\{X_1, X_2, \ldots, X_n\}$. What is the maximum likelihood estimator of the median of the distribution of Y?

(A) $\dfrac{\log 2}{M}$

(B) $M \log 2$

(C) $\log\left(\dfrac{2}{M}\right)$

(D) $\dfrac{n \log 2}{\sum_{i=1}^{n} X_i}$

24. Suppose X is a random variable with density

$$f_\theta(x) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$. The rejection region of the most powerful test for $H_0 : \theta = 4$ vs. $H_1 : \theta = 3$ at significance level 0.05 based on X is

(A) $\{X < (0.05)^{1/4}\}$. (B) $\{X > (0.95)^{1/3}\}$.

(C) $\{X < (0.05)^{1/3}\}$. (D) $\{X > (0.95)^{1/4}\}$.

25. A researcher collects data on the average daily air quality index (`AQI`) and average daily temperature (`Temp`) at a monitoring station for 365 consecutive days. She obtains the following least squares regression line with response `AQI` and predictor `Temp`.

$$\texttt{AQI} = 289 - 0.0049 \times \texttt{Temp}$$

The coefficient of determination R^2 of the model is 0.3844. What is the value of the sample correlation coefficient between `AQI` and `Temp`?

(A) -0.62 (B) 0.62 (C) -0.07 (D) 0.07

26. Suppose X_1 and X_2 are jointly distributed as Bivariate Normal with means μ_1, μ_2, variances σ_1^2, σ_2^2, and correlation ρ. Which of the following is equivalent to $H_0 : \sigma_1^2 = \sigma_2^2$?

    (A) X_1 and X_2 are independent.

    (B) X_1 and $X_2 - X_1$ are independent.

    (C) $X_1 + X_2$ and $X_1 - X_2$ are independent.

    (D) None of the above.

27. Given positive valued data $X_1, X_2, \ldots, X_n$, what is the value of θ that minimises

$$\lambda(\theta) = \sum_{i=1}^{n} \left| \frac{1}{X_i} - \frac{1}{\theta} \right| ?$$

    (A) Arithmetic mean of $X_1, X_2, \ldots, X_n$

    (B) Median of $X_1, X_2, \ldots, X_n$

    (C) Median of $\frac{1}{X_1}, \frac{1}{X_2}, \ldots, \frac{1}{X_n}$

    (D) Harmonic mean of $X_1, X_2, \ldots, X_n$

28. Suppose $X_1, X_2, \ldots, X_n$ are independent Poisson(θ) random variables with $\theta > 0$. Let $m \geq 1$ be an integer. An unbiased estimator of θ^m exists

    (A) only for odd m.

    (B) only for $m \leq n$.

    (C) only for $m \in \{1, 2\}$.

    (D) for all integers m.

29. Suppose $X_1, \ldots, X_n$ are i.i.d. $N(1, \sigma^2)$ for some unknown $\sigma^2 > 0$. What is the minimum variance unbiased estimator of σ^2?

(A) $\dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(X_i - 1)^2$

(B) $\dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n}(X_i - \overline{X})^2$, where $\overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$

(C) $\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(X_i^2 - 1)$

(D) $\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(X_i - 1)^2$

30. Suppose we have bivariate data (x_i, y_i), $i = 1, 2, \ldots, n$ with $n > 2$ such that $x_i \neq x_j$ and $y_i \neq y_j$ whenever $i \neq j$. Let r_1 be the sample Pearson's correlation coefficient, and r_2 be the sample Spearman's rank correlation coefficient between x and y. Consider the following statements.

I. If $|r_1| = 1$, then $r_2 = r_1$.

II. If $|r_2| = 1$, then $r_1 = r_2$.

III. If $r_1 < 0$, then $r_2 \leq 0$.

Then

(A) only II and III are correct.

(B) only I and III are correct.

(C) only I and II are correct.

(D) only I is correct.