

Notations and Abbreviations

The following are used throughout the question paper.

\mathbb{R}	Set of real numbers
\mathbb{E}	Expectation
\mathbb{V}	Variance
Cov	Covariance
$\lfloor x \rfloor$	Largest integer less than or equal to x
iid	independent and identically distributed
pmf	probability mass function
cdf	cumulative distribution function
MLE	Maximum likelihood estimator

1. Let n be an odd positive integer. For any permutation σ of $\{1, 2, 3, \dots, n\}$, consider the product

$$P(\sigma) = \prod_{j=1}^n (j - \sigma(j)) = (1 - \sigma(1))(2 - \sigma(2)) \cdots (n - \sigma(n)).$$

Then

- (A) $P(\sigma) = 0$ for all σ .
- (B) $P(\sigma)$ is odd for all σ .
- (C) $P(\sigma)$ is even for all σ .
- (D) None of the above.

2. Let a, b, c be three distinct non-zero real numbers. The expression

$$a \frac{(x-b)(x-c)}{(a-b)(a-c)} + b \frac{(x-c)(x-a)}{(b-c)(b-a)} + c \frac{(x-a)(x-b)}{(c-a)(c-b)} - x$$

is equal to zero for

- (A) exactly 2 values of x . (B) exactly 3 values of x .
(C) infinitely many values of x . (D) no value of x .
3. Equation of the straight line passing through the centres of the two circles $x^2 + y^2 - 4x = 0$ and $2x^2 + 2y^2 + 4x - 5y = 0$ is
- (A) $(x-2)y + (x-1)(y+5/4) = 0$.
(B) $4(x+1)y - (x-2)(4y-5) = 0$.
(C) $(x+1)y - (x-2)(y-5) = 0$.
(D) $2x + y = 3$.
4. Let p be a positive real and T be the set of all triangles that have perimeter p . A triangle in T has maximum area if its angles are

- (A) $60^\circ, 60^\circ$ and 60° (B) $90^\circ, 45^\circ$ and 45°
(C) $120^\circ, 30^\circ$ and 30° (D) $50^\circ, 60^\circ$ and 70°

5. For $x \geq 0$, let $f(x)$ denote the non-negative square root of x . The expression

$$f(\sin^4 x + 4 \cos^2 x) - f(\cos^4 x + 4 \sin^2 x)$$

equals

- (A) $\sin 2x$ (B) $\cos x$ (C) $\sin x$ (D) $\cos 2x$

6. Let S be the set of all 3×3 matrices A such that $A^2 = 0$ and the entries of A belong to $\{0, 1\}$. Then the number of elements in the set S is
- (A) 9 (B) 12 (C) 15 (D) 6
7. Let A be an $m \times n$ matrix and let P be an $n \times n$ idempotent matrix. If $P \neq I$, then which of the following statements is true?
- (A) $\text{rank}(AP) < m$
(B) $\text{rank}(AP) < \min\{m, n\}$
(C) $\text{rank}(AP) = \min\{m, n\}$
(D) $\text{rank}(AP) < n$
8. Let a_1, a_2, a_3, a_4 be four distinct positive numbers. Consider the 4×4 matrix $B = ((b_{ij}))$ where $b_{ij} = a_i + a_j$ for $i, j = 1, 2, 3, 4$. Then $\text{rank}(B)$ is
- (A) 1 (B) 4 (C) 2 (D) 3
9. Let S be the set of vectors $\mathbf{x} = (x_1, x_2, x_3)$ in \mathbb{R}^3 such that $x_i \in \{1, 2, 3, 4, 5\}$ for all i and not all x_i 's are equal. For how many ordered pairs $(\mathbf{x}, \mathbf{y}) \in S \times S$ are \mathbf{x} and \mathbf{y} linearly dependent vectors in \mathbb{R}^3 ?
- (A) 137 (B) 125 (C) 126 (D) 132

10. Let $f : [0, 1] \rightarrow [0, 1]$ be a function such that

$$|f(x) - f(y)| < |x - y|$$

for all $x \neq y \in [0, 1]$. Consider the following two statements:

- (i) There exists at least one point $a \in [0, 1]$ such that $f(a) = a$.
- (ii) There exists at most one point $a \in [0, 1]$ such that $f(a) = a$.

Choose the correct option.

- (A) Both (i) and (ii) are true.
- (B) (ii) is true but (i) is false.
- (C) (i) is true but (ii) is false.
- (D) Both (i) and (ii) are false.

11. The number of subsets $\{a, b, c\}$ of $\{1, 2, \dots, 24\}$ such that a, b and c are in arithmetic progression is

- (A) 66 (B) 132 (C) 276 (D) 138

12. Let f and g be two real-valued differentiable functions on \mathbb{R} . Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $h(x) = \max\{f(x), g(x)\}$. Consider the following two statements:

- (i) If h is differentiable then $f \geq g$ or $f \leq g$.
- (ii) If $f \geq g$ or $f \leq g$ then h is differentiable.

Choose the correct option.

- (A) (ii) is true but (i) is false.
- (B) h is always differentiable.
- (C) Both (i) and (ii) are true.
- (D) (i) is true but (ii) is false.

13. Let $X_{(1)} \leq \dots \leq X_{(n)}$ denote the order statistics from a random sample of size n , drawn from an exponential distribution with mean equal to 1. For $z > y > 0$, the conditional density of $X_{(n)}$ given $X_{(1)} = y$ is

- (A) $n \exp\{-(z - y)\} (1 - \exp\{-(z - y)\})^{n-1}$
- (B) $(n - 1) \exp\{-(z - y)\} (\exp\{-(z - y)\})^{n-2}$
- (C) $(n - 1) \exp\{-(z - y)\} (1 - \exp\{-(z - y)\})^{n-2}$
- (D) $\exp\{-(z - y)\}$

14. Let F and G be two cdfs and let $\alpha, \beta > 0$. Which of the following is not a cdf?

- (A) $(\alpha F(\alpha x) + \beta G(\beta x)) / (\alpha + \beta)$
- (B) $F(\alpha x)G(\beta x)$
- (C) $(F(\alpha x) + G(\beta x)) / (\alpha + \beta)$
- (D) $(F(\alpha x))^\alpha (G(\beta x))^\beta$

15. Suppose X_1, X_2 and X_3 are three random variables such that $X_1, (X_1 + X_2)/2$ and $(X_1 + X_2 + X_3)/3$ are iid standard normal random variables. Based on the above information choose the correct statement from the options given below.

- (A) This is not possible.
- (B) X_1 and X_3 are independent normal random variables.
- (C) X_1, X_2 and X_3 are identically distributed normal random variables.
- (D) X_1, X_2 and X_3 are independent normal random variables.

16. Suppose E_1, \dots, E_m are disjoint events satisfying $0 < \mathbb{P}(E_j) < 1$, for each $j = 1, \dots, m$. Let $E = \bigcup_{j=1}^m E_j$. Assume $0 < \mathbb{P}(E) < 1$. Let A be any other event and define $p = \mathbb{P}(A \mid E)$. Which of the following statements is true?

(A) $p = \sum_{j=1}^m \mathbb{P}(A \mid E_j) \mathbb{P}(E)$

(B) $p = \sum_{j=1}^m \mathbb{P}(A \mid E_j) \mathbb{P}(E_j \mid E)$

(C) $p = \sum_{j=1}^m \mathbb{P}(A \mid E_j)$

(D) $p = \sum_{j=1}^m \mathbb{P}(A \mid E_j) \mathbb{P}(E_j)$

17. Suppose A is the set of all 6-digit numbers formed using each of the digits $1, 2, 3, \dots, 6$ exactly once. If a number X is chosen uniformly at random from A , then what is the probability that X is divisible by 6 but not by 9 or 11?

(A) $1/2$ (B) $1/6$ (C) $1/11$ (D) $2/33$

18. Suppose X is a random variable taking values a^n with probability $(1-p)p^n$ for $n \geq 0$, where $a \neq 0$ is a real number and $0 < p < 1$. Consider the following statements.

(i) $\mathbb{E}[X]$ is finite for all values of a and p .

(ii) $\mathbb{V}(X)$ is finite whenever $|a| < 1/\sqrt{p}$.

Which of the following statements is correct?

- (A) Only (ii) is true.
 (B) Both (i) and (ii) are true.
 (C) Only (i) is true.
 (D) Neither (i) nor (ii) is true.

19. A coin with success probability $1/4$ is independently tossed N times and the outcomes are recorded. Given that the total number of successes is m , what is the probability that the first k tosses are successes, where $k \leq m$?

(A) $(1/4)^k(3/4)^{N-k}$ (B) $\frac{k}{m}$
(C) $\frac{m(m-1)\cdots(m-k+1)}{N(N-1)\cdots(N-k+1)}$ (D) $\binom{m}{k}(1/4)^k(3/4)^{m-k}$

20. Suppose X and Y are two independent random variables both following Poisson distribution with parameter 10. What is the value of $\mathbb{E}(X - Y)^2$?

(A) 30 (B) 10 (C) 40 (D) 20

21. Let M denote the region bounded by the parallelogram with vertices at $(0, 1)$, $(2, 1)$, $(0, -1)$ and $(-2, -1)$. Let (X, Y) denote the x and y co-ordinates of a randomly chosen point from M . Consider the following statements.

- (i) The marginal distribution of X is uniform on $(-2, 2)$.
- (ii) $\mathbb{E}(XY) = 0$.
- (iii) X and Y are independent.

Which of the following statements is correct?

- (A) Only (ii) and (iii) are true.
- (B) Only (ii) is true.
- (C) Only (i) is true.
- (D) None of the statements is true.

22. Suppose $\{X_i : i \geq 1\}$ are iid Poisson random variables with standard deviation 2. Then as $n \rightarrow \infty$

$$\frac{1}{n} \sum_{i=1}^n X_i(X_i - 2)$$

converges in probability to

- (A) 16 (B) 8 (C) 12 (D) 2

23. For $x_1, x_2, \dots, x_{2n} \in [-1, 1]$, where the x_i 's are not necessarily distinct, define

$$m = \frac{1}{2n} \sum_{i=1}^{2n} x_i \quad \text{and} \quad s^2 = \frac{1}{2n} \sum_{i=1}^{2n} (x_i - m)^2.$$

Let s_{\max}^2 denote the maximum possible value that s^2 can take over all such x_1, x_2, \dots, x_{2n} .

Based on the above information, choose the FALSE statement from the options given below.

- (A) $s^2 \in [0, 1]$ for all such x_1, x_2, \dots, x_{2n} .
(B) $m \in [-1, 1]$ for all such x_1, x_2, \dots, x_{2n} .
(C) If $m = 0$ then the corresponding $s^2 = s_{\max}^2$.
(D) If $s^2 = s_{\max}^2$ then the corresponding $m = 0$.

24. Consider a random variable with pmf given by

$$P(X = x) = \begin{cases} p & \text{if } x = 1, \\ 3p & \text{if } x = 2, \\ 2p & \text{if } x = 3, \\ 1 - 6p & \text{if } x = 4, \end{cases}$$

where the unknown parameter $p \in [1/20, 1/10]$. A random sample of size 4 from this distribution yielded the observations $x_1 = 2, x_2 = 3, x_3 = 2, x_4 = 1$. What is the maximum likelihood estimate of p based on these observations?

- (A) $1/10$ (B) $1/\sqrt[4]{18}$ (C) $1/18$ (D) $1/20$

25. Suppose that we have observations $(x_i, y_i, z_i), i = 1, \dots, n$. Let $\hat{\beta}_x$ be the estimated coefficient obtained by least-squares linear regression of z on x , without intercept. Let $\hat{\gamma}_x, \hat{\gamma}_y$ be the estimated coefficients obtained by similarly regressing z on x and y jointly, also without intercept. Which of the following statements is true?

- (A) $\sum_{i=1}^n (z_i - \hat{\beta}_x x_i)^2 > \sum_{i=1}^n (z_i - \hat{\gamma}_x x_i)^2$.
- (B) $\sum_{i=1}^n (z_i - \hat{\beta}_x x_i)^2 < \sum_{i=1}^n (z_i - \hat{\gamma}_x x_i - \hat{\gamma}_y y_i)^2$.
- (C) $\sum_{i=1}^n (z_i - \hat{\beta}_x x_i)^2 \leq \sum_{i=1}^n (z_i - \hat{\gamma}_y y_i)^2$.
- (D) $\sum_{i=1}^n (z_i - \hat{\beta}_x x_i)^2 \geq \sum_{i=1}^n (z_i - \hat{\gamma}_x x_i - \hat{\gamma}_y y_i)^2$.

26. Suppose that we have observations $(x_1, y_1), \dots, (x_n, y_n)$. We fit a simple linear regression of y on x , with intercept, using these observations via the least-squares method. Let $e_i, i = 1, \dots, n$, be the residuals of the fitted regression model. Consider the following statements.

(i) $\frac{1}{n} \sum_{i=1}^n e_i = 0$.

(ii) $\frac{1}{n} \sum_{i=1}^n (e_i - \bar{e})(x_i - \bar{x}) = 0$ where $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ and $\bar{e} = \frac{1}{n} \sum_{i=1}^n e_i$.

(iii) $\frac{1}{n} \sum_{i=1}^n e_i x_i = 0$.

Which of the following statements is correct?

- (A) (i), (ii) and (iii) are all true.
- (B) Only (ii) and (iii) are true.
- (C) Only (i) and (ii) are true.
- (D) Only (i) and (iii) are true.

27. Let X_1, \dots, X_n be iid $N(\mu, \sigma^2)$ random variables, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown parameters. Let $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ denote the cdf of $N(0, 1)$ and its inverse respectively. Define

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{and} \quad S_n^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

For $\alpha \in (0, 1)$, let $\xi_\alpha(\mu, \sigma)$ be a parametric function satisfying

$$\mathbb{P}(X_1 \leq \xi_\alpha(\mu, \sigma)) = \alpha.$$

What is the MLE of $\xi_\alpha(\mu, \sigma)$?

- (A) $\Phi(\bar{X}_n + \alpha S_n)$.
- (B) $\bar{X}_n + \alpha S_n$.
- (C) $X_{(\lfloor n\alpha \rfloor)}$, the $\lfloor n\alpha \rfloor$ -th order statistic of X_1, \dots, X_n .
- (D) $\bar{X}_n + \Phi^{-1}(\alpha) S_n$.

28. A random sample is obtained from a $N(\mu, 1)$ distribution, where $\mu \in \mathbb{R}$ is an unknown parameter. Using this sample, a most powerful (MP) test ϕ of size $\alpha = 1/2$ is constructed for testing $H_0 : \mu = 0$ against $H_1 : \mu = 1$. Let β denote the power of this test. Which of the following statements is correct?

- (A) ϕ is an MP test of size $< 1/2$ with power β for testing $H'_0 : \mu = -1$ against $H'_1 : \mu = 1$.
- (B) ϕ is an MP test of size $1/2$ with power $> \beta$ for testing $H'_0 : \mu = 0$ against $H'_1 : \mu = 1/2$.
- (C) $\psi = (1 - \phi)$ is not an MP test of size $(1 - \beta)$ for testing $H'_0 : \mu = 1$ against $H'_1 : \mu = 0$.
- (D) ϕ is an MP test of size $1/2$ for testing $H'_0 : \mu = 0$ against $H'_1 : \mu = a$ for some $a \in (-\infty, 0)$.

29. Let $\lambda > 0$. Let X_1, X_2, X_3 be independent Poisson random variables with parameters $\lambda, 2\lambda$ and $\lambda/2$ respectively. Which of the following statistics is sufficient for λ ?

- (A) $T = 2X_1 + 4X_2 + X_3$
- (B) $T = 2X_1 + X_2 + 4X_3$
- (C) $T = \max\{2X_1, 4X_2, X_3\}$
- (D) $T = X_1 + X_2 + X_3$

30. Let $N \geq 3$ be an integer. A box contains N balls numbered $\{1, 2, \dots, N\}$. A simple random sample s of size n ($1 < n < N$) is selected **with replacement** from the box. For $i = 1, \dots, N$, define

$$W_i = \begin{cases} 1 & \text{if the } i\text{-th ball is selected in } s, \\ 0 & \text{otherwise.} \end{cases}$$

Which of the following statements is true?

- (A) $\text{Cov}(W_1, W_2) < 0$.
- (B) $\sum_{i=1}^N W_i = n$.
- (C) $\mathbb{E}(W_i) = n/N$ for each $i = 1, \dots, N$.
- (D) $\text{Cov}(W_1, W_2) = 0$.