Answer all questions.

1. (a) Find $\lim_{x \to 0} \left(\frac{\sqrt{(1+x)} - \sqrt{(1+x^2)}}{\sqrt{(1-x^2)} - \sqrt{(1-x)}} \right)$.

   (b) Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ be two vectors in $\mathcal{R}^2$ such that $x_1y_1 + x_2y_2 = 0$ and $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1$. Prove that $B = \{\alpha, \beta\}$ is a basis for $\mathcal{R}^2$.

   [7 + 8 = 15]

2. (a) Find the maximum area of a triangle when the perimeter of the triangle is s.

   (b) In the quadratic equation, $x^2 + bx + c = 0$, both b and c can take any value from $1, 2, 3, 4, 5$ and 6. Find the probability that the equation will have real roots.

   (c) Pick a uniformly chosen random point inside a unit square (a square of side length 1) and draw a circle of radius $\frac{1}{3}$ around the point. Find the probability that the circle lies entirely inside the square.

   [5 + 5 + 5 = 15]

3. (a) Check the convergence of the following infinite series:

$$\frac{1}{4}+\frac{1\cdot 3}{4\cdot 7}+\frac{1\cdot 3\cdot 5}{4\cdot 7\cdot 10}+\frac{1\cdot 3\cdot 5\cdot 7}{4\cdot 7\cdot 10\cdot 13}+\cdots$$

Justify your answer.

(b) Let $a, b, c \in R$, such that $a^2+b^2+c^2=4$ and

$$A=\begin{bmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{bmatrix}.$$

Find the maximum and minimum values of the determinant of A.

$[6+9=15]$

4. (a) Find $\lim_{n\to\infty}\int_0^1 \log\left(1+\frac{x}{n}\right)dx$.

(b) An elevator starts at ground floor and can stop at any floor in between the 1^{st} and 10^{th}. Seven passengers are getting into the lift from the ground floor. What is the probability that no two passengers leave at the same floor?

$[7+8=15]$

5. (a) Sheena Enterprises and Geetha Enterprises are two industries that supply shock absorbers to a truck manufacturing company. The reliability of the shock absorbers from the respective industries is 95% and 90%. The truck manufacturer recently sold a batch of 100 trucks, with 60 trucks equipped with shock absorbers from Sheena Enterprises and 40 trucks equipped with shock absorbers from Geetha Enterprises. One of the truck owners has reported failure of one shock absorber. What is the probability that the failed shock absorber was supplied by Sheena Enterprises?

   (b) Consider a square $OBCD$ with length 2 units having side OB on the positive x-axis with O as the origin and C in the first quadrant. The square is rotated in an anti-clockwise direction to form a new square $OB'C'D'$ so that $\angle BOB' = 30°$. Find the coordinates of the square $OB'C'D'$.

$[5 + 10 = 15]$

6. (a) Prove that any perfect cube is the difference between two perfect squares.

   (b) Find all the conditions on a and b, belonging to the complex plane, for which $az + b\bar{z} + c = 0$ represents a straight line.

$[7 + 8 = 15]$

7. (a) A cleaning product manufacturer has three surface cleaning products: Shine-On, Shine-X, and Shine-Ultra. Shine-On contains 2% thickener and 1% foam enhancer. Shine-X contains 1.5% thickener and 2.5% foam enhancer. Shine-Ultra contains 1% thickener and 1.5% foam enhancer. The company wants to create a new surface cleaner by blending these three products so that the final mixture contains 1.6% thickener and 1.8% foam enhancer. There are 400 milliliters of Shine-On and 500 milliliters of Shine-X available, and the company needs to produce 1000 liters of the new cleaner. How much of Shine-On, Shine-X, and Shine-Ultra should the company mix to achieve the desired composition?

   (b) Find a point on the curve $y = (x-3)^2$ where the tangent is parallel to the line joining the points (4,1) and (3,0).

[8 + 7 = 15]

8. (a) Solve $\int_0^{\ln(1+\pi)} e^x \cos(1-e^x)dx$.

   (b) Show that for a Poisson distribution with unit mean, the mean deviation about mean is $\frac{2}{e}$ times standard deviation.

[7 + 8 = 15]