

1. (a) Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{x^5} \int_0^x e^{-t^2} dt - \frac{1}{x^4} + \frac{1}{3x^2} \right)$.

- (b) Given a circle of radius R , a square is inscribed in it. A circle is then inscribed in the square. Again a square in the circle and so on. Let A_i be the area of the i th square created in this process. Find the limit of $\sum_{i=1}^n A_i$ as $n \rightarrow \infty$.

[8 + 7 = 15]

2. (a) Find the total number of ways of forming two 5-digit numbers A and B such that B can be subtracted from A without using the method of borrowing at any stage.

- (b) If n is a positive integer, find the coefficient of x^{-1} in the expansion of $(1+x)^n \left(1 + \frac{1}{x}\right)^n$.

[8 + 7 = 15]

3. (a) Consider the following system of equations

$$\begin{bmatrix} 6 & p & 4 \\ 2 & 2 & -2 \\ 18 & -10 & 22 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r \\ -3 \\ 1 \end{bmatrix}$$

Find the values of p and r , so that the system of equations has infinitely many solutions.

- (b) Suppose U and W are distinct five-dimensional subspaces of a vector space V with $\dim(V) = 7$. Find the possible dimensions of $U \cap W$.

[7 + 8 = 15]

4. (a) Find the inverse function of $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$, $x \in \mathbb{R}$.
- (b) A cubic polynomial $P(x)$ contains only terms of the odd degrees. If $P(x)$ is divided by $(x - 3)$, then the remainder is 6. When $P(x)$ is divided by $(x^2 - 9)$, then the remainder is $g(x)$. Find the value of $g(2)$.

[9 + 6 = 15]

5. (a) A chemical is used to remove the impurities from the pulp by a bleaching process generating toxic and non-toxic wastes. The toxic waste generated is 0.05 times the square of the amount of chemical used, while the non-toxic waste generated is equal to 0.1 times the amount of chemical used. The pulp yield is 19.1 times the amount of chemical used. Determine the amount of chemical to be used to maximize the pulp yield and minimize the wastes.
- (b) A bank has implemented a fraud detection software. The probability that the software detect a fraudulent transaction as fraudulent is 0.99. The probability that the software detect a genuine transaction as genuine is 0.95. From the past experience, it is known that 0.01% of transactions are fraudulent. If the software detect a transaction as fraudulent, what is the probability that the transaction is actually fraudulent?

[8 + 7 = 15]

6. (a) Check the differentiability of the function

$$f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos |x|$$

at $x = 1$ and $x = 2$.

- (b) Find the point on the curve $3x^2 - 4y^2 = 72$ which is nearest to the line $3x + 2y + 1 = 0$.

[8 + 7 = 15]

7. (a) Show that for any integer n ,

$$n \left(\sqrt[n]{n+1} - 1 \right) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < n \left(1 - \frac{1}{\sqrt[n]{n}} \right) + 1.$$

- (b) Solve the equation for real x, y and z ,

$$\sqrt{x-1} + 2\sqrt{y-4} + 3\sqrt{z-9} = \frac{1}{2}(x+y+z).$$

[10 + 5 = 15]

8. (a) In an increasing GP, the sum of the first and the last term is 66. The product of the second term and the second from the last term is 128. The sum of all the terms is 126. Then find the number of terms in the GP.
- (b) Examine whether the following series is convergent

$$\sum_{n=1}^{\infty} \frac{n!}{1.3.5 \dots (2n-1)}.$$

[9 + 6 = 15]