MS(QMS)
2019

1. (a) Find the value of $\lim_{n\to\infty}(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{3n})$.
   (b) Show that every square matrix is uniquely expressible as the sum of a symmetric and a skew symmetric matrix. [7 + 8 = 15]

2. (a) Let x be a positive real number. Then show that for any x, $x^2 + \pi^2 + x^{2\pi} \geq x^\pi(\pi + x) + x\pi$.
   (b) Solve the differential equation $\frac{dy}{dx} = x - xy$. [10 + 5 = 15]

3. (a) Let x be chosen at random from the interval $(0, 1)$. What is the probability that $[\log_{10} 4x] - [\log_{10} x] = 0$? Here $[x]$ denotes the greatest integer that is less than or equal to x.
   (b) The graph of $2x^2+xy+3y^2-11x-20y+40 = 0$ is an ellipse in the first quadrant of the $xy-$plane. Let a and b be the maximum and minimum values of y/x over all the points (x, y) on the ellipse. Find the value of $a + b$. [8 + 7 = 15]

4. (a) Let $f(x)$ be the determinant of the following matrix:

$$\begin{bmatrix} a^2x+1 & (b^2-1)x & (c^2-1)x \\ (a^2-1)x & b^2x+1 & (c^2-1)x \\ (a^2-1)x & (b^2-1)x & c^2x+1 \end{bmatrix}.$$

   If $a^2 + b^2 + c^2 = 2$, then what is the degree of the polynomial $f(x)$?
   (b) Let $g(x) = x^6 - x^5 + x^2 - x + 3, -\infty < x < \infty$. Show that $g(x) > 0$ for all x. [7 + 8 = 15]

5. (a) Evaluate $\int\int \sqrt{\frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}}dxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
   (b) Let $f(x)$ and $g(x)$ are differentiable functions for $0 \leq x \leq 1$ such that $f(0) = 2, g(0) = 0, f(1) = 6$ and $g(1) = 2$. Then show that there exists c satisfying $0 < c < 1$ and $f'(c) = 2g'(c)$.
   [9 + 6 = 15]

6. (a) Let $\{x_n : n = 0, 1, 2, \ldots\}$ be a sequence of real numbers such that $x_{n+1} = \lambda x_n + (1-\lambda)x_{n-1}$ for $n \geq 1$ and for some $\lambda, 0 < \lambda < 1$.
      i. Show that $x_n = x_0 + (x_1 - x_0)\sum_{k=0}^{n-1}(\lambda - 1)^k$
      ii. Hence or otherwise show that x_n converges as $n \to \infty$ and find the limit.
   (b) Show that the height of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h is $\frac{h}{3}$. [(6 + 3) + 6 = 15]

7. (a) Let $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$. Then show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(x)dx = 1 - \frac{1}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{2}\log 2$.

(b) Let $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ be such that $\min f(x) > \max g(x)$. Show that $|c| > \sqrt{2}|b|$. [8 + 7 = 15]

8. (a) A piece of cheese is located at $(12, 10)$ in a co-ordinate plane. A mouse is at $(4, -2)$ and is running up the line $y = -5x + 18$. At the point (a, b) the mouse starts getting farther from the cheese rather than closer to it. What is $a + b$?

(b) Let $S_1 = \{(x, y) | \log_{10}(1 + x^2 + y^2) \leq 1 + \log_{10}(x + y)\}$ and $S_2 = \{(x, y) | \log_{10}(2 + x^2 + y^2) \leq 2 + \log_{10}(x + y)\}$. What is the ratio of the area of S_2 to the area of S_1? [5 + 10 = 15]