The symbol $\mathbb{R}$ will denote the set of all real numbers while $\mathbb{C}$ will denote the set of all complex numbers.

1. Let n, m and p be positive integers and suppose $g : \mathbb{R}^m \to \mathbb{R}^p$ be a linear map which is onto.

   (a) Prove that there exists a linear map $h : \mathbb{R}^p \to \mathbb{R}^m$ such that h is one-to-one and $g(h(x)) = x$ for all $x \in \mathbb{R}^p$.

   (b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map which is one-to-one and $\mathrm{Ran}(f) = \mathrm{Ker}(g)$. Prove that
   $$\mathrm{Ran}(f) \cap \mathrm{Ran}(h) = \{0\}.$$

   (c) Prove that
   $$\mathbb{R}^m = \{f(v) + h(x) : v \in \mathbb{R}^n, x \in \mathbb{R}^p\}.$$

2. Let V be a finite-dimensional real vector space and $T : V \to V$ a linear map which is diagonalizable. Let λ be an eigenvalue of T and $W = \mathrm{Ker}(T - \lambda I)$, where $I : V \to V$ is the identity map.

   (a) Show that there exists a subspace U of V such that
   $$U \cap W = \{0\} \quad \text{and} \quad V = \{u + w : u \in U, w \in W\}.$$

   (b) Suppose $S : V \to V$ is a linear map such that $ST = TS$. Prove that $S(U) \subseteq U$ and $S(W) \subseteq W$.

3. Let R be a commutative ring with unity. Suppose R has exactly two non-zero prime ideals P_1 and P_2. Assume that $P_1 \cap P_2 = (0)$.

   (a) Prove that P_1 and P_2 are maximal ideals.

   (b) Prove that R is a product of two fields.

   (c) Find all the ideals of the ring R.

4. Let S_3 be the symmetric group on three elements. Compute the number of group homomorphisms from S_3 to S_3.

5. Let $f : [0, 1) \to \mathbb{R}$ be a bounded continuous function. Define
   $$F(x) = \int_0^x f(t)\, dt, \quad x \in [0, 1).$$

   (a) Prove that $\lim\limits_{x \to 1^-} F(x)$ exists.

   (b) Define $F(1) = \lim\limits_{x \to 1^-} F(x)$. Suppose f is also monotonically increasing. Prove that $\lim\limits_{h \to 0^-} \frac{F(1+h) - F(1)}{h}$ exists.

6. Suppose $g : [0,1] \to \mathbb{R}$ is a continuous function and $f : [0,1] \to \mathbb{R}$ is a function satisfying the following conditions:

   (a) $f(0) > 0$,

   (b) $f(1) < 0$,

   (c) The function $h : [0,1] \to \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ is increasing.

   Let $A = \{x \in [0,1] : f(x) \geq 0\}$ and let $x_0 = \sup A$.

   (a) Prove that $f(x_0) \geq 0$.

   (b) Prove that there exists $t \in [x_0, 1]$ such that $g(t) = h(x_0)$.

   (c) Prove that $f(x_0) = 0$.

7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{N\to\infty} \int_0^N |f(x)|\, dx < \infty.$$

   (a) Show that

$$\lim_{N\to\infty} \frac{1}{N} \int_{\sqrt{N}}^N x f(x)\, dx = 0.$$

   (b) Using (a) or otherwise, show that

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N x f(x)\, dx = 0.$$

8. Let X and Y be metric spaces and $f : X \to Y$ be a function.

   (a) If f is continuous and K is a compact subset of X, prove that the function $f|_K : K \to Y$ defined by

$$f|_K(x) = f(x), x \in K,$$

   is continuous.

   (b) If $f|_K$ is continuous for all compact subsets K of X, then prove that f is continuous.