

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that f' is continuous, and there exist $a, b \in \mathbb{R}$ such that

$$\lim_{x \rightarrow \infty} f(x) = a \quad \text{and} \quad \lim_{x \rightarrow \infty} f'(x) = b.$$

Show that $b = 0$.

2. Suppose V is a finite dimensional real vector space, and

$$T : V \rightarrow V$$

is a linear transformation such that $\text{Rank}(T^2) = \text{Rank}(T)$.

- (a) Prove that $\text{Ker}(T) = \text{Ker}(T^2)$.
(b) Prove that $\text{Ker}(T) \cap T(V) = \{0\}$.

Here $T(V) = \{T(v) : v \in V\}$.

3. Let V be a finite dimensional real vector space and suppose P_1, P_2 are two non-zero linear transformations from V to V such that the following conditions hold:

- $P_1 + P_2 = I$, I is the identity map from V to V .
- $P_1P_2 = P_2P_1 = 0$.

Prove the following statements:

- (a) $P_i^2 = P_i$, for $i = 1, 2$.
(b) Let $i \in \{1, 2\}$. Then prove that the following statements are equivalent:
(i) $v \in P_i(V)$.
(ii) $P_iv = v$.
(c) $P_1(V) \cap P_2(V) = \{0\}$.

4. Let H be a normal subgroup of a group G such that there is a group homomorphism $\pi : G \rightarrow H$, with $\pi(h) = h$ for all $h \in H$. Prove that G is isomorphic to $H \times G/H$.

5. Let \mathbb{Z} denote the ring of integers, and

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}.$$

- (a) Prove that any maximal ideal of $\mathbb{Z}[\sqrt{-5}]$ contains a prime number $p \in \mathbb{Z}$.
- (b) Deduce that if M is a maximal ideal of $\mathbb{Z}[\sqrt{-5}]$, then $\frac{\mathbb{Z}[\sqrt{-5}]}{M}$ is a finite field.

6. Let $\{f_n\}_{n=1}^\infty$ be a sequence of real-valued continuous functions on \mathbb{R} such that

$$f_n\left(x + \frac{1}{n}\right) = f_n(x) \quad \text{for all } x \in \mathbb{R} \text{ and } n \geq 1.$$

Suppose there is a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f_n \rightarrow f$ uniformly on \mathbb{R} as $n \rightarrow \infty$. Show that f is a constant function.

(You may use the following fact: If $f_n \rightarrow f$ uniformly on \mathbb{R} and $x_n \rightarrow x$, then $f_n(x_n) \rightarrow f(x)$.)

7. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function.

(a) For $0 < a < 1$, show that

$$\lim_{n \rightarrow \infty} \int_0^a (n+1)x^n f(x) dx = 0.$$

(b) If $f(1) = 0$, then show that

$$\lim_{n \rightarrow \infty} \int_0^1 (n+1)x^n f(x) dx = 0.$$

8. Let (X, d) be a metric space. Let $B(X)$ denote the real vector space of bounded real-valued functions on X , that is,

$$B(X) = \left\{ f : X \rightarrow \mathbb{R} \mid \text{There exists } M \text{ (depending on } f) \text{ such that } |f(x)| \leq M \text{ for all } x \in X \right\}.$$

For $f \in B(X)$, define

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

Fix an $x_0 \in X$. For each $x \in X$, let $\phi_x : X \rightarrow \mathbb{R}$ be the function

$$\phi_x(y) = d(x, y) - d(x_0, y) \quad \text{for all } y \in X.$$

(a) Show that $\phi_x \in B(X)$ for all $x \in X$.

(b) Show that $\|\phi_x - \phi_y\| = d(x, y)$ for all $x, y \in X$.