The symbol $\mathbb{R}$ will denote the set of all real numbers while $\mathbb{C}$ will denote the set of all complex numbers. For a natural number m, $M_m(\mathbb{R})$ will denote the vector space of all $m \times m$ matrices with real entries and $GL_m(\mathbb{R})$ will denote the set of all $m \times m$ invertible matrices with real entries.

1. Let $A \in M_3(\mathbb{R})$ with eigenvalues $1, 2, 3$. Then
   (a) $A^{-1} = \frac{1}{6}(A^2 - 6A + 11I_3)$.
   (b) $A^{-1} = \frac{1}{6}(A^2 + 6A + 11I_3)$.
   (c) $A^{-1} = \frac{1}{6}(A^2 + 6A - 11I_3)$.
   (d) $A^{-1} = \frac{1}{6}(A^2 - 6A - 11I_3)$.

2. Let n and m be positive integers. Consider the vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ equipped with the Euclidean norms. For any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$T(\{x \in \mathbb{R}^n : ||x|| = 1\}) \subseteq \{y \in \mathbb{R}^m : ||y|| = 2\},$$

   which of the following statements must be true?
   (a) T is one-one.
   (b) T is onto.
   (c) $\text{Rank}(T) \geq m - n$.
   (d) $||T(x)|| \leq 3$ for all $x \in \mathbb{R}^n$.

3. Consider the matrix $A = \begin{pmatrix} 3 & 1 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

   Which of the following statements is correct?
   (a) There exists $B \in M_3(\mathbb{R})$ such that $B \neq 0$ and $BA = 0$.
   (b) $A^3 = 0$.
   (c) $\text{Rank}(A) = 0$.
   (d) $\dim \text{Ker}(A) = 2$.

4. Let A and B be similar 3×3 matrices with real entries. Which of the following statements is not necessarily correct?
   (a) If A is invertible, then B is invertible.
   (b) $\text{Rank}(A) = \text{Rank}(B)$.
   (c) $\text{Ker}(A) = \text{Ker}(B)$.
   (d) The characteristic polynomials of A and B are the same.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T^2 = 0$ but $T \neq 0$. Which of the following statements is not correct?
   (a) The only eigenvalue of T is zero.
   (b) For all $v \in \mathbb{R}^2$, either v is an eigenvector of T or $\{v, Tv\}$ is a basis of $\mathbb{R}^2$.
   (c) T is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.
   (d) T is diagonalizable.

6. For a $k \times k$ matrix A, A^t will denote the transpose of A and the (i,j)-th entry of A will be denoted by a_{ij}. Let V be the subspace of $M_8(\mathbb{R})$ defined as

$$V = \left\{ \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} : T \in M_3(\mathbb{R}), S \in M_5(\mathbb{R}), T^t = -T, s_{11} + s_{22} + s_{33} + s_{44} + s_{55} = 0 \right\}.$$

Then the dimension of the vector space V is

(a) 25.
(b) 26.
(c) 27.
(d) 28.

7. Let V denote the vector space of all polynomials with real coefficients of degree less than or equal to 5. Moreover, we define two linear maps S and T from V to V by

$$S(f) = f', T(f) = f''.$$

Which of the following statements is not correct?

(a) $\text{Rank}(S) = 5$.
(b) S and T are similar linear maps, i.e, there exists an invertible linear map $U : V \to V$ such that $U \circ S \circ U^{-1} = T$.
(c) 0 is the only eigenvalue of $S \circ T$.
(d) $S \circ T = T \circ S$.

8. The rank of a 4×4 real matrix with exactly 5 nonzero entries **cannot** be

(a) 1.
(b) 2.
(c) 3.
(d) 4.

9. Let us denote the subspace of $\mathbb{R}^4$ spanned by the vectors $(1,0,0,0)$ and $(0,1,0,0)$ by W. Moreover, let X denote the subspace of $\mathbb{R}^4$ defined as

$$X = \{x \in \mathbb{R}^4 : \langle x, y \rangle = 0 \text{ for all } y \in W\}.$$

Then

(a) dimension of X is 1.
(b) dimension of X is 2.
(c) dimension of X is 3.
(d) dimension of X is 4.

10. Let $R = \mathbb{R}[X,Y]/(XY-1)$ and I be an ideal of R generated by the image of the element $X - Y$ in R. Examine which of the following statements is correct.

(a) R/I is an integral domain with exactly two maximal ideals.
(b) R/I is a field.
(c) R/I is not an integral domain and R/I has exactly two maximal ideals.
(d) R/I has infinitely many maximal ideals.

11. Let I be an ideal of the polynomial ring $\mathbb{Z}[X]$ generated by the set $\{11X^{11}, 13X^7, 17X^5, 19X^3\}$. What is the minimum number of generators of I?

(a) 1.
(b) 2.
(c) 3.
(d) 4.

12. Let M be a fixed element of $GL_3(\mathbb{R})$ and S_3 the symmetric group on 3 elements. Suppose $\phi : S_3 \to GL_3(\mathbb{R})$ is a group homomorphism such that $M\phi(\sigma)M^{-1}$ is a diagonal matrix for all $\sigma \in S_3$. Which of the following statements is necessarily true?

(a) $\text{Ker}(\phi) = A_3$.
(b) $e^{\frac{2\pi i}{3}}$ is an eigenvalue of $\phi((123))$.
(c) $\phi((12)) = I_3$.
(d) $\phi((123)) = I_3$.

13. The number of cyclic subgroups of order 6 in S_5 is equal to

(a) 10.
(b) 20.
(c) 4.
(d) 5.

14. Let G be a group of order $2024 = 2^3.11.23$. Which of the following statements is necessarily correct?

(a) If G is abelian, then G is cyclic.
(b) Every subgroup H of G whose order is divisible by 23 has a normal cyclic subgroup K having more than one element.
(c) There does not exist any normal cyclic subgroup K of G containing more than one element.
(d) Suppose $K = G/H$ where H is a normal subgroup of G containing more than one element. Then K is cyclic.

15. Two ideals I and J in a ring R are called comaximal if there exist $x \in I$ and $y \in J$ such that $x+y=1$. Suppose I and J are two comaximal proper ideals in a ring R. Which of the following statements is not necessarily correct?

(a) I^2 and $I^2 + J^2$ are comaximal.
(b) I and J^2 are comaximal.
(c) $I + J^5$ and I are comaximal.
(d) $I^2 + JI$ and $I^2J + I^3$ are comaximal.

16. Let R be a finite commutative ring. Which of the following statements is necessarily correct?

(a) R is a field if it has a unique prime ideal.
(b) R is a product of fields.
(c) R is a field if the group of units of R is cyclic.
(d) If any degree 2 polynomial f in $R[x]$ has at most two roots in R, then R is a field.

17. Let N denote the number of elements of order 2 in a group of order 98. Which of the following statements is necessarily correct?

(a) $N \leq 2$.
(b) N can be more than 2 but N is less than or equal to 7.
(c) N can be more than 7 but $N \leq 49$.

(d) N can be greater than 49.

18. The value of the infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{k} \lim_{n\to\infty} \Big(\frac{1}{n^{k+1}} \sum_{m=1}^{n} m^k \Big)$$

is equal to

(a) 1.

(b) 2.

(c) e.

(d) 3.

19. The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$

(a) converges uniformly on $[-1, 1)$ to a function f and f is a continuous function on $[-1, 1)$.

(b) converges pointwise on $[-1, 1)$ to a function f but the convergence is not uniform.

(c) does not converge pointwise everywhere in $[-1, 1)$.

(d) converges uniformly on $(-1, 1)$ to a function f but f is not a continuous function on $[-1, 1)$.

20. Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series of real numbers. Which of the following statements must be correct?

(a) $\sum_{n=1}^{\infty} |a_n|$ converges.

(b) Both $\sum_{n=1}^{\infty} \frac{a_n}{n}$ and $\sum_{n=1}^{\infty} \frac{|a_n|}{n^2}$ converges.

(c) Both $\sum_{n=1}^{\infty} \frac{|a_n|}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

(d) $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ converges.

21. Let

$$S = \Big\{ x \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|x|^n}{1 + x^{2n}} \text{ is convergent} \Big\}.$$

Then

(a) $S = \mathbb{R}$.

(b) $S = \mathbb{R} \setminus \{-1, 1\}$.

(c) $S = \{x \in \mathbb{R} : |x| < 1\}$.

(d) $S = \{x \in \mathbb{R} : |x| > 1\}$.

22. For all continuous functions $f : \mathbb{R} \to \mathbb{R}$ and bounded sequences $\{x_n\}_{n=1}^{\infty}$, which of the following statements is correct?

(a) $\liminf_{n\to\infty} f(x_n) = f(\liminf_{n\to\infty} x_n)$.

(b) $\liminf_{n\to\infty} f(x_n) \geq f(\liminf_{n\to\infty} x_n)$.

(c) $\liminf_{n\to\infty} f(x_n) \leq f(\liminf_{n\to\infty} x_n)$.

(d) $\limsup_{n\to\infty} f(x_n) \leq f(\limsup_{n\to\infty} x_n)$.

23. Let $f : [0,1] \to \mathbb{R}$ be a monotonically increasing function such that $f(0) = 0$ and $f(1) = 100$. Let

$$A = \left\{ a \in (0,1) : \lim_{x \to a^+} f(x) - \lim_{x \to a^-} f(x) \geq \frac{1}{4} \right\}.$$

Which of the following may possibly be the cardinality of the set A?

(a) 500.
(b) 200.
(c) 1000.
(d) ∞.

24. Let $U \subseteq \mathbb{R}$ be an open set and $f : U \to \mathbb{R}$ be a differentiable function such that $f'(x) = 0$ for all $x \in U$. Let $A = f(U)$. Which of the following statements is necessarily correct?

(a) A must contain exactly one element.
(b) A may contain more than one but finitely many elements.
(c) A is either finite or countably infinite.
(d) A can be any subset of $\mathbb{R}$.

25. Let $S_k = \sum_{m=1}^{k} \frac{1}{m}$. Define a sequence $\{x_n\}_{n \geq 1}$ as

$$x_n = \sum_{k=2}^{n} \frac{1}{k S_{k-1} S_k}.$$

Then $\lim_{n \to \infty} x_n$ equals

(a) 0.
(b) 1.
(c) $\frac{1}{2}$.
(d) ∞.

26. For a function $f : \mathbb{R} \to \mathbb{R}$, we define two sets as follows:

$$A(f) = \{x \in [0,1] : f(x) = 0\} \text{ and } B(f) = \{x \in [0,1] : f'(x) = 0\}\}.$$

Let $\mathcal{C}$ denote the set of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(0) = f(1) = 0 \text{ and } A(f) \cap B(f) = \phi.$$

Which of the following statements is correct?

(a) For all $f \in \mathcal{C}$, $A(f)$ is a finite set.
(b) There exists $f \in \mathcal{C}$ such that $A(f)$ is infinite.
(c) For all $f \in \mathcal{C}$, $B(f)$ is a finite set.
(d) There exists $f \in \mathcal{C}$ such that $B(f) = \phi$.

27. A set $A \subseteq [-1,1]$ is said to have property P if for all Riemann integrable function $f : [-1,1] \to \mathbb{R}$, there exists $x \in A$ (x may depend on f) such that

$$\int_{-1}^{x} f(t)dt - \int_{x}^{1} f(t)dt = 0.$$

Which of the following statements is correct?

(a) $[-1,1]$ does not have property P.
(b) $[-1,1)$ does not have property P but $[-1,1]$ has property P.
(c) $(-1,1)$ does not have property P but $[-1,1)$ has property P.
(d) For all a,b such that $-1 < a < b < 1$, the set (a,b) does not have property P but $(-1,1)$ has property P.

28. A function $f : \mathbb{R} \to M_n(\mathbb{R})$ can be expressed as $f(t) = (a_{ij}(t)) \in M_n(\mathbb{R})$, where $a_{ij}(t)$ is the (i,j)-th entry of $f(t)$. The function f is said to be differentiable if each coordinate function a_{ij} is a (real valued) differentiable function on $\mathbb{R}$. Then its derivative $f' : \mathbb{R} \to M_n(\mathbb{R})$ is defined by $f'(t) = (a'_{ij}(t))$.

Let $f : \mathbb{R} \to M_2(\mathbb{R})$ be defined by

$$f(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Which of the following is false?

(a) $f'(\theta)f'(\theta)^t = I_2$.
(b) $f(\theta+\tau) = f(\theta)f(\tau)$.
(c) $f'(\theta)f'(\tau) = -f(\theta)f(\tau)$.
(d) $f'(\theta)f(\tau) = -f'(\tau)f(\theta)$.

29. Let X, Y be metric spaces and $f : X \to Y$ be a continuous function. Which of the following statements is not necessarily correct?

(a) If $\{x_n\}_n$ is a convergent sequence in X, then $\{f(x_n)\}_n$ is a convergent sequence in Y.
(b) If $\{x_n\}_n$ is a Cauchy sequence in X, then $\{f(x_n)\}_n$ is a Cauchy sequence in Y.
(c) If $\{x_n\}_n$ has a convergent subsequence in X, then $\{f(x_n)\}_n$ has a convergent subsequence in Y.
(d) If $\{x_n\}_n$ is dense in a compact subset of X, then $\{f(x_n)\}_n$ is dense in a compact subset of Y.

30. Let X and Y be subsets of $\mathbb{R}^2$ defined as

$$X = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \text{ and}$$
$$Y = (\{-1,1\} \times [-1,1]) \cup ([-1,1] \times \{-1,1\}).$$

Then X and Y are equipped with the metric d defined by

$$d((x,y),(x',y')) = \sqrt{(x-x')^2 + (y-y')^2}.$$

Which of the following statements is correct?

(a) There is a continuous one-one function $f : X \to Y$ which is not surjective.
(b) There is a function $f : X \to Y$ satisfying $d(f(x), f(y)) = d(x,y)$ for all $x, y \in X$.
(c) There is a bijective continuous function $f : X \to Y$ such that f^{-1} is also continuous.
(d) There is no non-constant continuous function from X to Y.

31. A subset A of $\mathbb{R}^2$ is said to be path-connected if given any two points x, y in A, there exists a path $\gamma : [0,1] \to A$ such that $\gamma(0) = x$ and $\gamma(1) = y$.
Consider a subset S of $\mathbb{R}^2$ which is the union of 4 circles of radius 1 having centres at

$$(1,0), (-1,0), (0,1) \text{ and } (0,-1).$$

Then

(a) $S \setminus \{(0,0)\}$ is not path-connected.
(b) $S \setminus \{(2,0),(0,0)\}$ is not path-connected.
(c) $S \setminus \{(2,0),(-2,0)\}$ is not path-connected.
(d) None of the above is true.