- $\mathbb{N}$ denotes the set of all positive integers.
- $\mathbb{Z}$ denotes the set of all integers.
- $\mathbb{R}$ denotes the set of all real numbers.
- For $n \in \mathbb{N}$, $M_n(\mathbb{R})$ denotes the set of all $n \times n$ matrices with real entries.
- For $A \in M_n(\mathbb{R})$, $\text{Ker}(A) = \{v \in \mathbb{R}^n : Av = 0\}$.
- For $n \in \mathbb{N}$, $GL_n(\mathbb{R})$ denotes the set of all $n \times n$ invertible matrices with real entries.
- For $n \in \mathbb{N}$, I_n denotes the $n \times n$ identity matrix.
- $\dim(V)$ denotes the dimension of a finite dimensional real vector space V.
- For a function $f : X \to Y$, $f(X)$ denotes the set $\{f(x) : x \in X\}$.
- For vector spaces V_1, V_2, V_3 and linear maps $S : V_1 \to V_2$ and $T : V_2 \to V_3$, the symbol $TS : V_1 \to V_3$ denotes the linear map defined as $TS(v) = T(S(v))$ for all $v \in V_1$.
- For $n \in \mathbb{N}$, S_n denotes the symmetric group on n elements.
- For $n \in \mathbb{N}$, A_n denotes the alternating group on n elements.

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1. Suppose $A \in M_3(\mathbb{R})$ has eigenvalues $1, 2, 3$. Then

   (A) $A^{-1} = \frac{1}{6}(A^2 - 6A - 11I_3)$.

   (B) $A^{-1} = \frac{1}{6}(A^2 + 6A + 11I_3)$.

   (C) $A^{-1} = \frac{1}{6}(A^2 + 6A - 11I_3)$.

   (D) $A^{-1} = \frac{1}{6}(A^2 - 6A + 11I_3)$.

2. Let n and m be positive integers. Consider the vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ equipped with the respective Euclidean norms. Which of the following statements must be true for every linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfying the condition

$$T\Big(\{x \in \mathbb{R}^n : \|x\| = 1\}\Big) \subseteq \{y \in \mathbb{R}^m : \|y\| = 2\}?$$

(A) T is one to one.

(B) T is onto.

(C) $\text{Rank}(T) \geq m - n$.

(D) $\|T(x)\| \leq 3$ for all $x \in \mathbb{R}^n$.

3. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 2, 3, 4)$ and $(4, 3, 2, 1)$. Let X denote the subspace of $\mathbb{R}^4$ defined as

$$X = \{x \in \mathbb{R}^4 : \langle x, y\rangle = 0 \text{ for all } y \in W\},$$

where $\langle x, y\rangle = \sum_{i=1}^{4} x_i y_i$, for $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$. Then the dimension of X is equal to

(A) 1. (B) 2. (C) 3. (D) 4.

4. Consider the matrix $A = \begin{pmatrix} 3 & 1 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Which of the following statements is correct?

(A) There exists $B \in M_3(\mathbb{R})$ such that $B \neq 0$ and $BA = 0$.

(B) $A^3 = 0$.

(C) $\text{Rank}(A) = 0$.

(D) $\dim \text{Ker}(A) = 2$.

5. Suppose A and B are similar 3×3 matrices with real entries. Which of the following statements is **not** necessarily correct?

(A) If A is invertible, then B is invertible.

(B) $\text{Rank}(A) = \text{Rank}(B)$.

(C) $\text{Ker}(A) = \text{Ker}(B)$.

(D) The characteristic polynomial of A and B are the same.

6. For $A \in M_k(\mathbb{R})$, A^t denotes the transpose of A. Let V be the subspace of $M_8(\mathbb{R})$ consisting of matrices of the form $\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$ satisfying the following conditions:

(i) $T \in M_3(\mathbb{R}), S \in M_5(\mathbb{R})$,

(ii) $T^t = -T$,

(iii) the sum of the diagonal entries of S is equal to 0.

Then the dimension of the vector space V is

(A) 25. (B) 26. (C) 27. (D) 28.

7. Suppose $T \in M_2(\mathbb{R})$ is such that $T^2 = 0$ but $T \neq 0$. Which of the following statements is **not** correct?

(A) The only eigenvalue of T is zero.

(B) For all $v \in \mathbb{R}^2 \setminus \{0\}$, either v is an eigenvector of T or $\{v, Tv\}$ is a basis of $\mathbb{R}^2$.

(C) T is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

(D) T is diagonalizable.

8. Let V be the vector space of all polynomials with real coefficients of degree less than or equal to 5. Define two linear maps S and T from V to V by

$$S(f) = f', \quad T(f) = f''.$$

Which of the following statements is **not** correct?

(A) $\text{Rank}(S) = 5$.

(B) $ST = TS$.

(C) 0 is the only eigenvalue of ST.

(D) There exists an invertible linear map $U : V \to V$ such that $USU^{-1} = T$.

9. The rank of a 4×4 real matrix with exactly 5 nonzero entries **cannot** be

(A) 1. (B) 2. (C) 3. (D) 4.

10. The number of cyclic subgroups of order 6 in S_5 is equal to

(A) 10. (B) 20. (C) 4. (D) 5.

11. Let $R = \mathbb{R}[X, Y]/(XY - 1)$ and I be the ideal of R generated by the image of the element $X - Y$ in R. Which of the following statements is correct?

(A) R/I is an integral domain with exactly two maximal ideals.

(B) R/I is a field.

(C) R/I is not an integral domain and R/I has exactly two maximal ideals.

(D) R/I has infinitely many maximal ideals.

12. Let I be the ideal of the polynomial ring $\mathbb{Z}[X]$ generated by the set $\{11X^{11}, 13X^7, 17X^5, 19X^3\}$. What is the minimum number of generators of I?

(A) 1. (B) 2. (C) 3. (D) 4.

13. Let R be a finite commutative ring with unity. Which of the following statements is necessarily correct?

(A) R is a field if it has a unique prime ideal.

(B) R is a product of fields.

(C) R is a field if the group of units of R is cyclic.

(D) If any degree 2 polynomial f in $R[X]$ has at most two roots in R, then R is a field.

14. Let G be a group of order $2024 = 2^3 \times 11 \times 23$. Which of the following statements is necessarily correct?

(A) If G is abelian, then G is cyclic.

(B) Suppose $K = G/H$ where H is a normal subgroup of G containing more than one element. Then K is cyclic.

(C) There does not exist any normal cyclic subgroup K of G containing more than one element.

(D) Every subgroup H of G whose order is divisible by 23 has a normal cyclic subgroup K having more than one element.

15. Let N denote the number of elements of order 2 in a group of order 98. Which of the following statements is necessarily correct?

(A) $N \leq 2$.

(B) N can be more than 2 but N is less than or equal to 7.

(C) N can be more than 7 but $N \leq 49$.

(D) N can be greater than 49.

16. Let R be a commutative ring with unity. Two ideals I and J in R are called comaximal if there exist $x \in I$ and $y \in J$ such that $x + y = 1$. For two comaximal proper ideals I and J in R, which of the following statements is **not** correct?

(A) I^2 and $I^2 + J^2$ are comaximal.

(B) I and J^2 are comaximal.

(C) $I + J^5$ and I are comaximal.

(D) $I^2 + JI$ and $I^2J + I^3$ are comaximal.

17. Let M be a fixed element of $GL_3(\mathbb{R})$ and $\phi : S_3 \to GL_3(\mathbb{R})$ is a group homomorphism such that $M\phi(\sigma)M^{-1}$ is a diagonal matrix for all $\sigma \in S_3$. Which of the following statements is necessarily true?

(A) $\text{Ker}(\phi) = A_3$.

(B) $e^{\frac{2\pi i}{3}}$ is an eigenvalue of $\phi((123))$.

(C) $\phi((12)) = I_3$.

(D) $\phi((123)) = I_3$.

18. Which of the following statements is correct for every continuous function $f : \mathbb{R} \to \mathbb{R}$ and every bounded sequence $\{x_n\}$ of real numbers?

(A) $\liminf_{n\to\infty} f(x_n) = f(\liminf_{n\to\infty} x_n)$.

(B) $\liminf_{n\to\infty} f(x_n) \geq f(\liminf_{n\to\infty} x_n)$.

(C) $\liminf_{n\to\infty} f(x_n) \leq f(\liminf_{n\to\infty} x_n)$.

(D) $\limsup_{n\to\infty} f(x_n) \leq f(\limsup_{n\to\infty} x_n)$.

19. Let

$$S = \{x \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|x|^n}{1 + x^{2n}} \text{ is convergent}\}.$$

Then

(A) $S = \mathbb{R}$.

(B) $S = \mathbb{R} \setminus \{-1, 1\}$.

(C) $S = \{x \in \mathbb{R} : |x| < 1\}$.

(D) $S = \{x \in \mathbb{R} : |x| > 1\}$.

20. Let $S_k = \sum_{m=1}^{k} \frac{1}{m}$, $k \geq 1$. Define a sequence $\{x_n\}$, $n \geq 2$ as

$$x_n = \sum_{k=2}^{n} \frac{1}{kS_{k-1}S_k}.$$

Then $\lim_{n\to\infty} x_n$ equals

(A) 2. (B) 1. (C) $\frac{1}{2}$. (D) ∞.

21. The value of the infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{k} \lim_{n\to\infty} \left(\frac{1}{n^{k+1}} \sum_{m=1}^{n} m^k \right)$$

is equal to

(A) 1. (B) 2. (C) e. (D) 3.

22. Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series of real numbers. Which of the following statements must be correct?

(A) $\sum_{n=1}^{\infty} |a_n|$ converges.

(B) Both $\sum_{n=1}^{\infty} \frac{a_n}{n}$ and $\sum_{n=1}^{\infty} \frac{|a_n|}{n^2}$ converges.

(C) Both $\sum_{n=1}^{\infty} \frac{|a_n|}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

(D) $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ converges.

23. The series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

(A) converges uniformly on $[-1, 1)$ to a function f which is continuous on $[-1, 1)$.

(B) converges pointwise on $[-1, 1)$ to a function f but the convergence is not uniform.

(C) does not converge pointwise everywhere in $[-1, 1)$.

(D) converges uniformly on $(-1, 1)$ to a function f but f is not a continuous function on $[-1, 1)$.

24. Let $U \subseteq \mathbb{R}$ be an open set and $f : U \to \mathbb{R}$ be a differentiable function such that $f'(x) = 0$ for all $x \in U$. Let $A = f(U)$. Which of the following statements is necessarily correct?

(A) A must contain exactly one element.

(B) A may contain more than one but finitely many elements.

(C) A is either finite or countably infinite.

(D) A can be any subset of $\mathbb{R}$.

25. Let $f : [0, 1] \to \mathbb{R}$ be a monotonically increasing function such that $f(0) = 0$ and $f(1) = 100$. Let

$$A = \left\{ a \in (0, 1) : \lim_{x \to a+} f(x) - \lim_{x \to a-} f(x) \geq \frac{1}{4} \right\}.$$

Which of the following can be the cardinality of the set A?

(A) 500. (B) 200. (C) 1000. (D) ∞.

26. For a function $f : \mathbb{R} \to \mathbb{R}$, we define two sets as follows:

$A(f) = \{x \in [0, 1] : f(x) = 0\}$ and $B(f) = \{x \in [0, 1] : f'(x) = 0\}$.

Let $\mathcal{C}$ be the set of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(0) = f(1) = 0 \text{ and the sets } A(f), B(f) \text{ are disjoint.}$$

Which of the following statements is correct?

(A) For all $f \in \mathcal{C}$, $A(f)$ is a finite set.

(B) There exists $f \in \mathcal{C}$ such that $A(f)$ is infinite.

(C) For all $f \in \mathcal{C}$, $B(f)$ is a finite set.

(D) There exists $f \in \mathcal{C}$ such that $B(f)$ is empty.

27. A set $A \subseteq [-1,1]$ is said to have property P if for all Riemann integrable function $f : [-1,1] \to \mathbb{R}$, there exists $x \in A$ (x may depend on f) such that

$$\int_{-1}^{x} f(t)dt - \int_{x}^{1} f(t)dt = 0.$$

Which of the following statements is correct?

(A) $[-1,1]$ does not have property P.

(B) $[-1,1)$ does not have property P but $[-1,1]$ has property P.

(C) $(-1,1)$ does not have property P but $[-1,1)$ has property P.

(D) For all a, b such that $-1 < a < b < 1$, the set (a,b) does not have property P but $(-1,1)$ has property P.

28. Let X, Y be metric spaces and $f : X \to Y$ be a continuous function. Which of the following statements is **not** necessarily correct?

(A) If $\{x_n\}$ is a convergent sequence in X, then $\{f(x_n)\}$ is a convergent sequence in Y.

(B) If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.

(C) If $\{x_n\}$ has a convergent subsequence in X, then $\{f(x_n)\}$ has a convergent subsequence in Y.

(D) If $\{x_n\}$ is dense in a compact subset of X, then $\{f(x_n)\}$ is dense in a compact subset of Y.

29. Consider $\mathbb{R}^2$ equipped with the metric defined by

$$d((x,y),(x',y')) = \sqrt{(x-x')^2 + (y-y')^2}.$$

Let X and Y be subsets of $\mathbb{R}^2$ defined as

$$X = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \text{ and}$$

$$Y = \Big(\{-1,1\} \times [-1,1]\Big) \cup \Big([-1,1] \times \{-1,1\}\Big).$$

Which of the following statements is correct?

(A) There is a bijective continuous function $f : X \to Y$ such that f^{-1} is also continuous.

(B) There is a function $f : X \to Y$ satisfying $d(f(x), f(y)) = d(x,y)$ for all $x, y \in X$.

(C) There is a continuous one to one function $f : X \to Y$ which is not onto.

(D) There is no non-constant continuous function from X to Y.

30. An urn contains 50 red balls, 50 black balls and 100 white balls. If 80 balls are drawn from the urn without replacement, what is the probability that at least one ball of each color is drawn?

(A) $1 - \dfrac{2\binom{150}{80}}{\binom{200}{80}}$.

(B) $1 - \dfrac{\binom{150}{80}}{\binom{200}{80}}$.

(C) $1 - \dfrac{2\binom{150}{80}}{\binom{200}{80}} - \dfrac{\binom{100}{80}}{\binom{200}{80}}$.

(D) $1 - \dfrac{2\binom{150}{80}}{\binom{200}{80}} + \dfrac{\binom{100}{80}}{\binom{200}{80}}$.