- $\mathbb{N}$ denotes the set of all positive integers.
- $\mathbb{Z}$ denotes the set of all integers.
- $\mathbb{R}$ denotes the set of all real numbers.
- $M_n(\mathbb{R})$ denotes the set of all $n \times n$ matrices with real entries.
- I_n denotes the $n \times n$ identity matrix.
- $\dim(V)$ denotes the dimension of a finite dimensional vector space V.
- For a function $f : X \to Y$, $f(X)$ denotes the set $\{f(x) : x \in X\}$.

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1. Let V be the vector space over $\mathbb{R}$ consisting of all polynomials with real coefficients. Consider the subset W of all polynomials of degree 4. Which of the following statements is correct?

   (A) W is not a subspace of V.
   (B) W is a subspace of V having dimension 5.
   (C) W is a subspace of V having dimension 4.
   (D) W is a subspace of V having dimension 3.

2. If A belongs to $M_3(\mathbb{R})$, then which of the following statements is not true in general?

   (A) If A is diagonalizable, then A^2 is diagonalizable.
   (B) If A^2 is diagonalizable, then A is diagonalizable.
   (C) There exists a polynomial p of degree 3 such that $p(A) = 0$.
   (D) A has a real eigenvalue.

3. Let W denote the vector space over $\mathbb{R}$ consisting of all polynomials of degree less than or equal to 6 with real coefficients. Let $D : W \to W$ be the linear transformation defined by

$$D(p) = \frac{dp}{dx}.$$

Which of the following statements is correct?

(A) D is invertible.

(B) Rank of D is equal to 6.

(C) Rank of D is equal to 5.

(D) Rank of D is equal to 1.

4. Let V be a finite dimensional vector space and $P : V \to V$ be a non-zero linear transformation such that $P^2 = P$. Which of the following statements is not true in general?

(A) $\mathrm{Ker}\,(P) \cap \mathrm{Ker}\,(I - P) = \{0\}$.

(B) $(I - P)^2 = I - P$.

(C) $\mathrm{Ker}\,(P) = (I - P)(V)$.

(D) $\dim\big(\mathrm{Ker}\,(P)\big) = \dim\big(P(V)\big)$.

5. Let A be a 2×2 real matrix such that $(1, 0)$ is an eigenvector of A with eigenvalue $\log 2$, and $(0, 1)$ is an eigenvector of A with eigenvalue $\log 3$. Let I denote the 2×2 identity matrix. Then the limit in $\mathbb{R}^2$ of the vectors

$$\lim_{n\to\infty} \left(I + \frac{A}{n}\right)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(A) exists and equals $(2, 3)$.

(B) does not exist.

(C) exists and equals $(3, 5)$.

(D) exists and equals $(2, 5)$.

6. Consider the following vector spaces over $\mathbb{R}$:

$$\begin{aligned} V &= \{f : [0,1] \to \mathbb{R} \mid f \text{ is a continuous function}\}, \\ W &= \{f : [0,1] \to \mathbb{R} \mid f \text{ is a function}\}. \end{aligned}$$

If $T : V \to W$ is the map defined by

$$(Tf)(t) = tf(t), \quad t \in [0,1],$$

then which of the following statements is not true?

(A) $T(V) \subseteq V$.

(B) T is a linear map from V to V.

(C) T is one-to-one.

(D) $T(V) = V$.

7. Let $W = \{A \in M_4(\mathbb{R}) : \text{Rank}\,(A) \leq 3\}$. Which of the following statements is correct?

(A) W is not closed under scalar multiplication.

(B) If B, C belong to W, then $B + C$ belongs to W.

(C) If $B \in W$, then $I_4 + B \in W$.

(D) W is not a subspace of $M_4(\mathbb{R})$.

8. For positive integers r and n with $r < n$, let P denote the $n \times n$ matrix

$$\begin{bmatrix} I_r & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{bmatrix},$$

where $0_{k\times l}$ denotes the zero matrix with k rows and l columns. Let H^t denote the transpose of a matrix H and

$$V = \{H \in M_n(\mathbb{R}) : H = HP + PH = H^t\}.$$

Which of the following statements is true?

(A) $\dim\,(V) = r^2$.

(B) $\dim\,(V) = (n-r)^2$.

(C) $\dim\,(V) = r(n-r)$.

(D) $\dim\,(V) = 2r(n-r)$.

9. Let $\phi : G \to H$ be a group homomorphism between finite groups. Which of the following statements is not true in general?

(A) Image of any subgroup of G under ϕ is a subgroup of H.

(B) For every $g \in G$, order of $\phi(g)$ divides the order of g.

(C) $\phi(G)$ is a subgroup of H.

(D) Image of any normal subgroup of G under ϕ is a normal subgroup of H.

10. Consider the group $G = \mathbb{Z}_6 \times \mathbb{Z}_{15} \times \mathbb{Z}_{20}$. The order of the element $(1, 2, 3)$ in G is

(A) 30. (B) 40. (C) 60. (D) 120.

11. Consider the ring

$$\mathbb{Z}[\sqrt{-2022}] = \{a + b\sqrt{-2022} : a, b \in \mathbb{Z}\}.$$

The number of elements in this ring having multiplicative inverse is

(A) 2. (B) 3. (C) 4. (D) infinite.

12. Let p be a prime number. Consider the ring $R = \mathbb{Z}[x]/(x^2 - px)$ with multiplicative identity denoted by 1_R. Then, the number of ring homomorphisms $\phi : R \to \mathbb{Z}$ such that $\phi(1_R) = 1$ is

(A) 0. (B) 1. (C) 2. (D) 3.

13. Let R be a commutative ring with unity and $a, b \in R$. Consider the ideal

$$I = \langle a, b \rangle = \{xa + yb : x \in R, y \in R\}.$$

If

$$I^2 = \Big\{\sum_{i=1}^{n} a_i b_i : a_i, b_i \in I, n \in \mathbb{N}\Big\},$$

then which of the following statements is not true in general?

(A) $I = \langle a - 5b, b \rangle$.
(B) $I^2 = \langle a^2, b^2 \rangle$.
(C) $I^2 \subseteq I$.
(D) $I + I = I$.

14. For a prime number p and $k \in \mathbb{N}$, let $\overline{k}$ denote the equivalence class of k in the field $\mathbb{Z}_p$. Consider the following system of linear equations in $\mathbb{Z}_p$:

$$\begin{aligned} \overline{10}X + \overline{5}Y &= \overline{8} \\ \overline{3}X + \overline{12}Y &= \overline{11}. \end{aligned}$$

For which of the following values of p does the above system have a unique solution?

(A) $p = 3$.
(B) $p = 5$.
(C) $p = 7$.
(D) $p = 11$.

15. Let G be a group of order 75 which has an element of order 25. Then the number of elements of order 5 in G is

(A) 4.
(B) 5.
(C) 12.
(D) 24.

16. Let G be a group of order 30. Which of the following statements is necessarily true?

(A) G is abelian but may not be cyclic.

(B) G is cyclic.

(C) G cannot have more than 7 subgroups of order 5.

(D) G can simultaneously have one subgroup of order 2, five subgroups of order 3, and six subgroups of order 5.

17. The total number of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is

(A) 4. (B) 12. (C) 16. (D) 22.

18. If

$$a_n = \frac{\big(3 + (-1)^n\big)^n}{4^n},$$

then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges if and only if

(A) $x = 0$. (B) $x \in [-1, 1)$.
(C) $x \in (-1, 1]$. (D) $x \in (-1, 1)$.

19. If

$$\lim_{n\to\infty} \left(\frac{2^{2n} + 3^{2n}}{6^n} \right)^{1/n} = L,$$

then L is equal to

(A) $\frac{4}{3}$. (B) $\frac{13}{12}$. (C) $\frac{7}{6}$. (D) $\frac{3}{2}$.

20. For any $a \in [0,1]$, the limit of the sequence $\{x_n\}_{n=1}^{\infty}$, where

$$x_n = \frac{1 + 2^a + \ldots + n^a}{n^{a+1}}$$

is

(A) $1 - \frac{a}{2}$.

(B) $\frac{1}{1+a^2}$.

(C) $\frac{1}{1+a}$.

(D) $\frac{1}{2a^2 - a + 1}$.

21. Consider the sequence $\{f_n\}_{n=1}^{\infty}$ of functions on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ given by

$$f_n(x) = (\sin x)^n \cos x.$$

The sequence $\{f_n\}$ converges uniformly

(A) on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ but not on any strictly larger subset of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(B) on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ but not on any strictly larger subset of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(C) on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ but not on any strictly larger subset of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(D) on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

22. Let $a_n = \frac{n}{2n+1} \sin\left(\frac{2\pi}{3} n\right)$, $n \in \mathbb{N}$. Which of the following statements is true?

(A) $\lim_{n\to\infty} a_n = \frac{1}{2}$.

(B) $\limsup_{n\to\infty} a_n = \frac{\sqrt{3}}{4}$.

(C) $\limsup_{n\to\infty} a_n = \frac{\sqrt{3}}{2}$.

(D) $\limsup_{n\to\infty} a_n = 0$.

23. Let f and g be polynomials with real coefficients of degree m and n, respectively. Suppose $g(x) \neq 0$ for all $x \in [1, \infty)$. The integral

$$\int_1^\infty \frac{|f(x)|}{|g(x)|}\, dx$$

converges if and only if

(A) $n - m \geq 2$.

(B) $m - n \leq 2$.

(C) $n - m \geq 1$.

(D) $m \geq 1$ and $n \geq 1$.

24. Fix $x_0, x_1 \in \mathbb{R}$ with $x_0 \neq x_1$. Define

$$x_{n+1} = ax_n + (1-a)x_{n-1}, \quad n \in \mathbb{N}.$$

The sequence $\{x_n\}_{n=1}^\infty$ is convergent if and only if

(A) $a \in (0, 2)$.

(B) $a \in [0, 2)$.

(C) $a \in [0, 2]$.

(D) $a \in (0, 1)$.

25. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is continuous and satisfies

$$|f'(x) - e^{2x}| \leq 3 \quad \text{for all } x \in \mathbb{R}.$$

Then the limit

$$\lim_{x \to +\infty} \frac{f(x)}{e^{2x}}$$

(A) may not exist.

(B) exists and equals $\frac{1}{2}$.

(C) exists and equals $\frac{3}{4}$.

(D) exists and equals $\frac{1}{4}$.

26. If

$$\lim_{\epsilon \to 0^+} \int_{\epsilon}^{2\epsilon} \frac{e^{(x-1)^2}}{x} dx = L,$$

then L is equal to

(A) 2.

(B) $e \log 2$.

(C) $\log(e+2)$.

(D) e^2.

27. The following series of functions

$$\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}, \quad x \in \mathbb{R},$$

converges uniformly on

(A) $\mathbb{R} \setminus (a, b)$ for all $a \geq 0$.

(B) intervals of the form $(-\infty, a)$ and (a, ∞) for all $a \in \mathbb{R}$.

(C) $\mathbb{R} \setminus (a, b)$ for all $a < 0$ and $b > 0$.

(D) all compact subsets of $\mathbb{R}$.

28. Let (X, d) be a metric space and $x_0 \in X$. Let $\gamma : [0, \infty) \to X$ be such that

$$d\big(\gamma(s), \gamma(t)\big) = |s - t| \quad \text{for all } s, t \in [0, \infty).$$

Define a function $f : [0, \infty) \to \mathbb{R}$ by

$$f(t) = d\big(x_0, \gamma(t)\big) - d\big(\gamma(0), \gamma(t)\big), \quad t \in [0, \infty).$$

Which of the following statements is necessarily true?

(A) The function f is monotone increasing and bounded.

(B) The function f is monotone increasing and unbounded.

(C) The function f is monotone decreasing and bounded.

(D) The function f is monotone decreasing and unbounded.

29. Let (X, d) be a compact metric space. Let $x \in X$ be such that $X \setminus \{x\} = \{y \in X : y \neq x\}$ is complete. Which of the following statements is necessarily true?

   (A) There is a sequence $\{x_n\}_{n=1}^{\infty}$ in X converging to x such that $d(x_n, x) > 0$ for all $n \in \mathbb{N}$.

   (B) There exists $p \in X$ such that $0 < d(p, x) = \inf_{q \in X \setminus \{x\}} d(q, x)$.

   (C) Given a point $p \in X \setminus \{x\}$, there is a point $q \neq p$ such that $d(p, x) = d(q, x)$.

   (D) For every $p \in X \setminus \{x\}$, there is a point $q \in X$ such that $0 < d(q, x) < d(p, x)$.

30. The probability that a person tests positive for Covid by RTPCR, given that the person is infected with Covid, is 0.95. Also, the probability that a person tests positive for Covid by RTPCR, given that the person is not infected with Covid, is 0.15. Assume that in a country, 10% of the population is infected with Covid. A person is randomly chosen and the RTPCR test yields positive result. The probability that the person is actually infected with Covid is

   (A) $\frac{19}{20}$. (B) $\frac{17}{46}$. (C) $\frac{19}{46}$. (D) $\frac{9}{10}$.