

- \mathbb{N} denotes the set of all positive integers.
 - \mathbb{Z} denotes the set of all integers.
 - \mathbb{R} denotes the set of all real numbers.
 - $M_n(\mathbb{R})$ denotes the set of all $n \times n$ matrices with real entries.
 - I_n denotes the $n \times n$ identity matrix.
 - $\dim(V)$ denotes the dimension of a finite dimensional vector space V .
 - For a function $f : X \rightarrow Y$, $f(X)$ denotes the set $\{f(x) : x \in X\}$.
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1. If A belongs to $M_3(\mathbb{R})$, then which of the following statements is not true in general?
 - (A) If A is diagonalizable, then A^2 is diagonalizable.
 - (B) If A^2 is diagonalizable, then A is diagonalizable.
 - (C) There exists a polynomial p of degree 3 such that $p(A) = 0$.
 - (D) A has a real eigenvalue.
2. Let V be the vector space over \mathbb{R} consisting of all polynomials with real coefficients. Consider the subset W of all polynomials of degree 4. Which of the following statements is correct?
 - (A) W is not a subspace of V .
 - (B) W is a subspace of V having dimension 5.
 - (C) W is a subspace of V having dimension 4.
 - (D) W is a subspace of V having dimension 3.

3. Let V be a finite dimensional vector space and $P : V \rightarrow V$ be a non-zero linear transformation such that $P^2 = P$. Which of the following statements is not true in general?

- (A) $\text{Ker}(P) \cap \text{Ker}(I - P) = \{0\}$.
- (B) $(I - P)^2 = I - P$.
- (C) $\text{Ker}(P) = (I - P)(V)$.
- (D) $\dim(\text{Ker}(P)) = \dim(P(V))$.

4. Let W denote the vector space over \mathbb{R} consisting of all polynomials of degree less than or equal to 6 with real coefficients. Let $D : W \rightarrow W$ be the linear transformation defined by

$$D(p) = \frac{dp}{dx}.$$

Which of the following statements is correct?

- (A) D is invertible.
- (B) Rank of D is equal to 6.
- (C) Rank of D is equal to 5.
- (D) Rank of D is equal to 1.

5. Let A be a 2×2 real matrix such that $(1, 0)$ is an eigenvector of A with eigenvalue $\log 2$, and $(0, 1)$ is an eigenvector of A with eigenvalue $\log 3$. Let I denote the 2×2 identity matrix. Then the limit in \mathbb{R}^2 of the vectors

$$\lim_{n \rightarrow \infty} \left(I + \frac{A}{n} \right)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- (A) exists and equals $(2, 3)$.
- (B) does not exist.
- (C) exists and equals $(3, 5)$.
- (D) exists and equals $(2, 5)$.

6. Let $W = \{A \in M_4(\mathbb{R}) : \text{Rank}(A) \leq 3\}$. Which of the following statements is correct?

- (A) W is not closed under scalar multiplication.
- (B) If B, C belong to W , then $B + C$ belongs to W .
- (C) If $B \in W$, then $I_4 + B \in W$.
- (D) W is not a subspace of $M_4(\mathbb{R})$.

7. Consider the following vector spaces over \mathbb{R} :

$$V = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is a continuous function}\},$$

$$W = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is a function}\}.$$

If $T : V \rightarrow W$ is the map defined by

$$(Tf)(t) = tf(t), \quad t \in [0, 1],$$

then which of the following statements is not true?

- (A) $T(V) \subseteq V$.
- (B) T is a linear map from V to V .
- (C) T is one-to-one.
- (D) $T(V) = V$.

8. For positive integers r and n with $r < n$, let P denote the $n \times n$ matrix

$$\begin{bmatrix} I_r & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix},$$

where $0_{k \times l}$ denotes the zero matrix with k rows and l columns.

Let H^t denote the transpose of a matrix H and

$$V = \{H \in M_n(\mathbb{R}) : H = HP + PH = H^t\}.$$

Which of the following statements is true?

- (A) $\dim(V) = r^2$.
- (B) $\dim(V) = (n - r)^2$.
- (C) $\dim(V) = r(n - r)$.
- (D) $\dim(V) = 2r(n - r)$.

9. Consider the group $G = \mathbb{Z}_6 \times \mathbb{Z}_{15} \times \mathbb{Z}_{20}$. The order of the element $(1, 2, 3)$ in G is
- (A) 30. (B) 40. (C) 60. (D) 120.

10. Let $\phi : G \rightarrow H$ be a group homomorphism between finite groups. Which of the following statements is not true in general?
- (A) Image of any subgroup of G under ϕ is a subgroup of H .
(B) For every $g \in G$, order of $\phi(g)$ divides the order of g .
(C) $\phi(G)$ is a subgroup of H .
(D) Image of any normal subgroup of G under ϕ is a normal subgroup of H .

11. Let p be a prime number. Consider the ring $R = \mathbb{Z}[x]/(x^2 - px)$ with multiplicative identity denoted by 1_R . Then, the number of ring homomorphisms $\phi : R \rightarrow \mathbb{Z}$ such that $\phi(1_R) = 1$ is
- (A) 0. (B) 1. (C) 2. (D) 3.

12. Consider the ring

$$\mathbb{Z}[\sqrt{-2022}] = \{a + b\sqrt{-2022} : a, b \in \mathbb{Z}\}.$$

The number of elements in this ring having multiplicative inverse is

- (A) 2. (B) 3. (C) 4. (D) infinite.

13. For a prime number p and $k \in \mathbb{N}$, let \bar{k} denote the equivalence class of k in the field \mathbb{Z}_p . Consider the following system of linear equations in \mathbb{Z}_p :

$$\begin{aligned}\bar{10}X + \bar{5}Y &= \bar{8} \\ \bar{3}X + \bar{12}Y &= \bar{11}.\end{aligned}$$

For which of the following values of p does the above system have a unique solution?

- (A) $p = 3$. (B) $p = 5$. (C) $p = 7$. (D) $p = 11$.

14. Let R be a commutative ring with unity and $a, b \in R$. Consider the ideal

$$I = \langle a, b \rangle = \{xa + yb : x \in R, y \in R\}.$$

If

$$I^2 = \left\{ \sum_{i=1}^n a_i b_i : a_i, b_i \in I, n \in \mathbb{N} \right\},$$

then which of the following statements is not true in general?

- (A) $I = \langle a - 5b, b \rangle$. (B) $I^2 = \langle a^2, b^2 \rangle$.
(C) $I^2 \subseteq I$. (D) $I + I = I$.

15. Let G be a group of order 30. Which of the following statements is necessarily true?

- (A) G is abelian but may not be cyclic.
(B) G is cyclic.
(C) G cannot have more than 7 subgroups of order 5.
(D) G can simultaneously have one subgroup of order 2, five subgroups of order 3, and six subgroups of order 5.

21. For any $a \in [0, 1]$, the limit of the sequence $\{x_n\}_{n=1}^{\infty}$, where

$$x_n = \frac{1 + 2^a + \dots + n^a}{n^{a+1}}$$

is

(A) $1 - \frac{a}{2}$.

(B) $\frac{1}{1 + a^2}$.

(C) $\frac{1}{1 + a}$.

(D) $\frac{1}{2a^2 - a + 1}$.

22. Let f and g be polynomials with real coefficients of degree m and n , respectively. Suppose $g(x) \neq 0$ for all $x \in [1, \infty)$. The integral

$$\int_1^{\infty} \frac{|f(x)|}{|g(x)|} dx$$

converges if and only if

(A) $n - m \geq 2$.

(B) $m - n \leq 2$.

(C) $n - m \geq 1$.

(D) $m \geq 1$ and $n \geq 1$.

23. Let $a_n = \frac{n}{2n+1} \sin\left(\frac{2\pi}{3}n\right)$, $n \in \mathbb{N}$. Which of the following statements is true?

(A) $\lim_{n \rightarrow \infty} a_n = \frac{1}{2}$.

(B) $\limsup_{n \rightarrow \infty} a_n = \frac{\sqrt{3}}{4}$.

(C) $\limsup_{n \rightarrow \infty} a_n = \frac{\sqrt{3}}{2}$.

(D) $\limsup_{n \rightarrow \infty} a_n = 0$.

27. The following series of functions

$$\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}, \quad x \in \mathbb{R},$$

converges uniformly on

- (A) $\mathbb{R} \setminus (a, b)$ for all $a \geq 0$.
 - (B) intervals of the form $(-\infty, a)$ and (a, ∞) for all $a \in \mathbb{R}$.
 - (C) $\mathbb{R} \setminus (a, b)$ for all $a < 0$ and $b > 0$.
 - (D) all compact subsets of \mathbb{R} .
28. Let (X, d) be a compact metric space. Let $x \in X$ be such that $X \setminus \{x\} = \{y \in X : y \neq x\}$ is complete. Which of the following statements is necessarily true?
- (A) There is a sequence $\{x_n\}_{n=1}^{\infty}$ in X converging to x such that $d(x_n, x) > 0$ for all $n \in \mathbb{N}$.
 - (B) There exists $p \in X$ such that $0 < d(p, x) = \inf_{q \in X \setminus \{x\}} d(q, x)$.
 - (C) Given a point $p \in X \setminus \{x\}$, there is a point $q \neq p$ such that $d(p, x) = d(q, x)$.
 - (D) For every $p \in X \setminus \{x\}$, there is a point $q \in X$ such that $0 < d(q, x) < d(p, x)$.

29. Let (X, d) be a metric space and $x_0 \in X$. Let $\gamma : [0, \infty) \rightarrow X$ be such that

$$d(\gamma(s), \gamma(t)) = |s - t| \quad \text{for all } s, t \in [0, \infty).$$

Define a function $f : [0, \infty) \rightarrow \mathbb{R}$ by

$$f(t) = d(x_0, \gamma(t)) - d(\gamma(0), \gamma(t)), \quad t \in [0, \infty).$$

Which of the following statements is necessarily true?

- (A) The function f is monotone increasing and bounded.
 - (B) The function f is monotone increasing and unbounded.
 - (C) The function f is monotone decreasing and bounded.
 - (D) The function f is monotone decreasing and unbounded.
30. The probability that a person tests positive for Covid by RTPCR, given that the person is infected with Covid, is 0.95. Also, the probability that a person tests positive for Covid by RTPCR, given that the person is not infected with Covid, is 0.15. Assume that in a country, 10% of the population is infected with Covid. A person is randomly chosen and the RTPCR test yields positive result. The probability that the person is actually infected with Covid is

- (A) $\frac{19}{20}$. (B) $\frac{17}{46}$. (C) $\frac{19}{46}$. (D) $\frac{9}{10}$.
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