Notation. In the following, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \dots\}$. For an element g of a group G, $o(g)$ denotes the order of g in G. For a set A, $|A|$ denotes its cardinality.

(1) Let V be a finite dimensional vector space and let W be a proper subspace of V. Let W' be another subspace of V such that $V = W \oplus W'$, i.e. $V = \mathrm{Span}(W \cup W')$ and $W \cap W' = \{0\}$. Let $T : V \to V$ be an invertible linear map such that $T(W) \subset W$. Which of the following statements is necessarily true?

(a) $T(W') \subset W'$.
(b) $W' \subset T(W')$.
(c) $T(W') \cap W = \{0\}$.
(d) $W' \subset \mathrm{Ker}(T)$.

(2) Let V be a finite dimensional real vector space, and let $T : V \to V$ be a linear map such that $\mathrm{Range}(T) = \mathrm{Ker}(T)$. Which of the following statements is not necessarily true?

(a) $T = 0$.
(b) $T^2 = 0$.
(c) 0 is an eigenvalue of T.
(d) All eigenvalues of T are equal to 0.

(3) Consider the vector space $\mathbb{R}^n$ equipped with the Euclidean metric d defined by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

Let W be a proper subspace of $\mathbb{R}^n$. Which of the following statements is necessarily true?

(a) W is closed.
(b) W is open.
(c) W is not closed.
(d) W is neither closed nor open.

(4) Let A be a 5×5 real matrix. If $A = (a_{ij})$, let A_{ij} denote the cofactor of the entry a_{ij}, for $1 \le i, j \le 5$. Let $\hat{A}$ denote the matrix whose (i, j)-th entry is A_{ij}, $1 \le i, j \le 5$. Suppose the rank of A is 3. What is the rank of $\hat{A}$?

(a) 1
(b) 3
(c) 5
(d) 0

(5) For $n \geq 2$, the determinant of the $n \times n$ permutation matrix

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 1 & \ldots & \ldots & \ldots \\ 1 & 0 & \ldots & \ldots & \ldots \end{pmatrix}$$

is

(a) $(-1)^n$
(b) $(-1)^{n(n-1)/2}$
(c) -1
(d) 1

(6) Let $M_2(\mathbb{R})$ denote the vector space of all 2×2 matrices over the field of real numbers, i.e.

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.$$

Let $S \subset M_2(\mathbb{R})$ be the subspace defined by

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : a + c = 0 \right\}.$$

Then the dimension of S is

(a) 1
(b) 2
(c) 3
(d) 4

(7) Let V be a finite dimensional real vector space of dimension $n > 1$ and let $W \subset V$ be a subspace of dimension $n-1$. A linear map from V to $\mathbb{R}$ is called a linear functional on V. Which of the following statements is necessarily true?

(a) There does not exist any linear functional on V such that W is the kernel of that linear functional.
(b) W is the kernel of a unique linear functional on V.
(c) W is the kernel of a linear functional on V.
(d) There exists a non-zero linear functional on V whose kernel strictly contains W.

(8) Let $< .,. >_n$ denote the standard inner product on the vector space $\mathbb{R}^n$, i.e.

$$< x, y >_n = \sum_{i=1}^{n} x_i y_i$$

for vectors $x, y \in \mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map such that

$$< Tx, Ty >_m = < x, y >_n$$

for all $x, y \in \mathbb{R}^n$. Which of the following statements is necessarily true?

(a) $n \geq m$.
(b) $n \leq m$.
(c) $n = m$.
(d) The map T is onto.

(9) Let V be a finite dimensional real inner product space, and let $T : V \to V$ be a linear map such that $< Tx, Ty > = < x, y >$ for all $x, y \in V$. Suppose $W \subset V$ is a proper subspace of V such that $T(W) \subset W$. Define a subspace $W^\perp$ of V by

$$W^\perp := \{v \in V | \ < v, w >= 0 \text{ for all } w \in W\}.$$

Which of the following statements is necessarily true?

(a) $T(W^\perp)$ is not contained in $W^\perp$.
(b) $T(W^\perp)$ is contained in $W^\perp$.
(c) $T(W^\perp) \cap W^\perp = \{0\}$.
(d) $W^\perp$ is contained in $T(W^\perp)$.

(10) Let R be a ring with unit such that $a^2 = a$ for all $a \in R$. Which of the following statements is not necessarily true?

(a) $ab = -ba$ for all $a, b \in R$.
(b) $a = -a$ for all $a \in R$.
(c) R is commutative.
(d) $R = \{0, 1\}$.

(11) Let S be a nonempty set, and let $P(S)$ be the power set of S, i.e. $P(S) = \{A|A \subset S\}$. Define a binary operation Δ on $P(S)$ by $A\Delta B := (A\cup B)\backslash(A\cap B)$ for $A, B \in P(S)$. Which of the following statements is necessarily true?

(a) $(P(S), \Delta)$ is not a group as Δ is not associative.
(b) $(P(S), \Delta)$ is not a group as there is no identity.
(c) $(P(S), \Delta)$ is an abelian group.
(d) $(P(S), \Delta)$ is a non-abelian group.

(12) Let I_1, I_2 be ideals of a commutative ring R. Define the set

$$I_1 + I_2 := \{a + b | a \in I_1, b \in I_2\}.$$

Which of the following statements is not necessarily true?

(a) $I_1 + I_2$ is an ideal of R.
(b) $I_1 \subset I_1 + I_2$.
(c) $|I_1 + I_2| = |I_1| + |I_2|$ if R is finite.
(d) $|I_1 + I_2| = |I_1| \cdot |I_2|$ if $I_1 \cap I_2 = \{0\}$ and R is finite.

(13) Let I be an ideal of a commutative ring R. Define the set

$$\sqrt{I} := \{a \in R | \text{ There exists } n \geq 1 \text{ such that } a^n \in I\}.$$

Which of the following statements is necessarily true?

(a) $\sqrt{I}$ is an ideal.
(b) $\sqrt{I}$ is not an ideal.
(c) $\sqrt{I} = I$.
(d) $\sqrt{I} \subset I$.

(14) How many non-isomorphic groups are there of order 15?

(a) 1
(b) 2
(c) 3
(d) 5

(15) Suppose G is a group and $a, b \in G$. Which of the following statements is necessarily true?

(a) There exists a positive integer n such that $a^n = b^n$.
(b) $(ab)^{-1} = a^{-1}b^{-1}$.
(c) $o(ab) = o(ba)$.
(d) None of the above statements is necessarily true.

(16) Let H_1, H_2 be distinct subgroups of a finite abelian group G. Define the subgroup H_1H_2 by $H_1H_2 = \{h_1h_2 | h_1 \in H_1, h_2 \in H_2\}$.
    Which of the following statements is necessarily true?

(a) $|G| \leq |H_1| + |H_2|$.
(b) $|G/(H_1 \cap H_2)| = |G/H_1| \cdot |G/H_2|$.
(c) $|G/H_1| = |G/H_2| \cdot |H_2/(H_1 \cap H_2)|$.
(d) $|(H_1H_2)/H_1| = |H_2/(H_1 \cap H_2)|$.

(17) Let $p > 3$ be a prime number and let S_p denote the symmetric group on p symbols. How many p-Sylow subgroups are there in S_p?

(a) 1.
(b) p.
(c) 2.
(d) $(p - 2)!$.

(18) Let R be a commutative ring with unit and let

$$N = \{a \in R|\ a^n = 0 \text{ for some integer } n \geq 0\}.$$

Which of the following statements is necessarily true?

(a) Any prime ideal of R contains N.
(b) N is not an ideal.
(c) N is a prime ideal.
(d) $N = \{0\}$.

(19) Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable, and suppose $\lim_{x \to \infty} f''(x) = 1$. Which of the following statements is necessarily true?

(a) $\lim_{x \to \infty} \frac{f(x)}{x^2} = 1$.
(b) $\lim_{x \to \infty} \frac{f(x)}{x^2}$ does not exist.
(c) $\lim_{x \to \infty} \frac{f(x)}{x^2} = 2$.
(d) $\lim_{x \to \infty} \frac{f(x)}{x^2} = 1/2$.

(20) Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = (\cos(\pi x))^{2n}$. Which of the following statements is true?

(a) The sequence $\{f_n\}$ converges uniformly on $[0, 1]$.
(b) The sequence $\{f_n\}$ converges pointwise on $[0, 1]$ to a function f such that f has exactly one point of discontinuity.
(c) The sequence $\{f_n\}$ converges pointwise on $[0, 1]$ to a function f such that f has exactly two points of discontinuity.
(d) The sequence $\{f_n\}$ does not converge pointwise on $[0, 1]$.

(21) Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_{-1}^{1} f(x)x^{2n}dx = 0$$

for all $n \geq 0$. Which of the following statements is necessarily false?

(a) $\int_{-1}^{1} f(x)^2 \, dx = \int_{-1}^{1} f(-x)^2 \, dx$.
(b) $\left(\sup_{x \in [-1,1]} f(x)\right) + \left(\inf_{x \in [-1,1]} f(x)\right) = 0$.
(c) $f(0) \neq 0$.
(d) $f(1/2)f(-1/2) \leq 0$.

(22) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + 1) = f(x) + 1$ for all $x \in \mathbb{R}$. Which of the following statements is necessarily false?

(a) $\lim_{x \to \infty} \frac{f(x)}{x^{1+\epsilon}} = 0$ for all $\epsilon > 0$.
(b) $\lim_{x \to \infty} \frac{f(x)}{x}$ does not exist.
(c) $\lim_{x \to \infty} \frac{f(x)}{x} = 1$.
(d) $\lim_{x \to \infty} \frac{f(x)}{x^{1-\epsilon}} = +\infty$ for all $\epsilon > 0$.

(23) Let $\{0, 1\}^{\mathbb{N}}$ denote the set of all sequences $\{x_n\}$ such that $x_n \in \{0, 1\}$ for all $n \geq 1$. Define a map $f : \{0, 1\}^{\mathbb{N}} \to \mathbb{R}$ by

$$f(\{x_n\}) := \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Which of the following statements is true?

(a) The map f is one-to-one and onto from $\{0, 1\}^{\mathbb{N}}$ to $[0, 1]$.
(b) The map f is one-to-one and onto from $\{0, 1\}^{\mathbb{N}}$ to $[0, 1)$.
(c) The map f is onto from $\{0, 1\}^{\mathbb{N}}$ to $[0, 1]$ and $|f^{-1}(1/2)| = 2$.
(d) The map f is onto from $\{0, 1\}^{\mathbb{N}}$ to $[0, 1]$ and $|f^{-1}(1)| = 2$.

(24) Let $f : [0, \infty) \to \mathbb{R}$ be a monotone increasing function, and define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = f(x + n), \ x \in [0, \infty)$$

for all $n \geq 1$. Suppose that for some $x_0 \in [0, \infty)$, the limit $\lim_{n \to \infty} f_n(x_0)$ exists. Which of the following statements is necessarily false?

(a) The sequence $\{f_n\}$ converges pointwise on $[0, \infty)$.
(b) The sequence $\{f_n\}$ converges uniformly on $[0, \infty)$.
(c) The limit $\lim_{x \to \infty} f(x)$ exists.
(d) The function f is unbounded on $[0, \infty)$.

6

(25) Let X, Y be sets and let $f : X \to Y$ be a function. Let $\{S_i\}_{i \in I}$ be a family of subsets of X, i.e. $S_i \subset X$ for all $i \in I$, where I is an index set. Which of the following statements is not necessarily true?

(a) $f(\cup_{i \in I} S_i) \subset \cup_{i \in I} f(S_i)$.
(b) $f(\cup_{i \in I} S_i) \supset \cup_{i \in I} f(S_i)$.
(c) $f(\cap_{i \in I} S_i) \subset \cap_{i \in I} f(S_i)$.
(d) $f(\cap_{i \in I} S_i) \supset \cap_{i \in I} f(S_i)$.

(26) Let S be the set of all those nonnegative real numbers α with the following property: if $\{x_n\}$ is a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} x_n < +\infty$, then we also have $\sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n^\alpha} < +\infty$. Which of the following statements is true?

(a) $S = \emptyset$.
(b) $S \supset (1/4, \infty)$.
(c) $S \supset (1/2, \infty)$.
(d) $S \subset (3/4, \infty)$.

(27) Let X be a finite set. Let $P(X)$ be the power set of X, i.e. the set whose elements are all subsets of X. Which of the following defines a metric on the power set $P(X)$?

(a) $d(V, W) = |(V \cup W) \setminus (V \cap W)|$.
(b) $d(V, W) = |V \cap W|$.
(c) $d(V, W) = |V \setminus W|$.
(d) $d(V, W) = |V \cup W|$.

(28) The tangent line to the curve $2x^6 + y^4 = 9xy$ at the point $(1, 2)$ has slope

(a) $3/23$.
(b) $6/23$.
(c) $9/23$.
(d) $4/7$.

(29) Consider the following statements:

(i) If $\sum_n a_n$ and $\sum_n b_n$ are convergent, then $\sum_n a_n b_n$ is convergent.

(ii) If $\sum_n a_n$ is convergent and $\sum_n b_n$ is absolutely convergent, then $\sum_n a_n b_n$ is absolutely convergent.

(iii) If $a_n \geq 0$ for all n, $\sum_n a_n$ is convergent, and $\{b_n\}$ is a bounded sequence, then $\sum_n a_n b_n$ is absolutely convergent.

Which of the following statements is true?

(a) All of the statements (i), (ii), (iii) are true.
(b) Statement (ii) is true but statement (iii) is false.
(c) Only statements (i) and (iii) are true.
(d) Only statements (ii) and (iii) are true.

(30) Consider the metric space $(\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}, d)$, where the metric d is defined by $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$ for $m, n \in \mathbb{N}$, and $d(n, \infty) = 1/n$ for $n \in \mathbb{N}$. Let $f : \overline{\mathbb{N}} \to \mathbb{R}$ be a continuous function between metric spaces (where $\mathbb{R}$ is equipped with its usual metric). Which of the following statements is necessarily false?

(a) The metric space $\overline{\mathbb{N}}$ is compact.
(b) The function f is unbounded.
(c) The function f is uniformly continuous.
(d) For any $x \in \mathbb{R}$, the set $f^{-1}(\{x\})$ is compact.