

1. Let A_1 and A_2 be two arbitrarily chosen subsets of $\{1, 2, \dots, n\}$.
The number of ways such that $A_1 \cap A_2 = \emptyset$ is

(A) 2^n (B) $4^n - 3^n$ (C) 3^n (D) $3^n - 1$

2. The limit

$$\lim_{n \rightarrow \infty} \frac{1 + \sqrt{2} + \sqrt[3]{3} + \dots + \sqrt[n]{n}}{n}$$

(A) is equal to 0 (B) is equal to 1
(C) is equal to 2 (D) does not exist

3. For $x, y \geq 0$, consider the regions $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
and $B = \{(x, y) \in \mathbb{R}^2 : x^4 + y^4 > 1\}$. Then

(A) $A \subsetneq B$ (B) $B \subsetneq A$
(C) $A \cap B = \emptyset$ (D) $A \cup B = \mathbb{R}^2$

4. The roots of the equation $x^4 + x^3 + x^2 + x + 1 = 0$, when plotted
in the complex plane, are

(A) the vertices of a square
(B) some of the vertices of a regular pentagon
(C) some of the vertices of a regular hexagon
(D) some of the vertices of a regular heptagon

5. Let $f, g : [0, 1] \rightarrow (0, \infty)$ be two continuously differentiable functions such that $f'(x) = g'(x)$ for all $x \in [0, 1]$. Suppose that

$$\frac{f(\frac{1}{2})}{g(\frac{1}{2})} = 1. \text{ Then } \frac{f(\frac{1}{4})}{g(\frac{1}{4})} \text{ is equal to}$$

- (A) 1 (B) 2
(C) $\frac{1}{2}$ (D) either 2 or $\frac{1}{4}$

6. For $w \in \mathbb{C}$, let $\operatorname{Re} w$ and $\operatorname{Im} w$ denote the real and imaginary parts of w , respectively. Suppose $z \in \mathbb{C}$ is such that $|z| = 1$. Which of the following is necessarily true?

- (A) $\operatorname{Re} z$ is rational implies that $\operatorname{Re} z^2$ is rational
(B) $\operatorname{Re} z^2$ is rational implies that $\operatorname{Re} z$ is rational
(C) $\operatorname{Im} z$ is rational implies that $\operatorname{Im} z^2$ is rational
(D) $\operatorname{Im} z^2$ is rational implies that $\operatorname{Im} z$ is rational

7. The maximum value of the function

$$f(x) = \frac{x}{2} - \frac{1}{1+x^2}$$

on the set $\{x \in \mathbb{R} : x(x-1) \leq 12\}$ is

- (A) 0 (B) $\frac{33}{17}$ (C) $-\frac{8}{5}$ (D) -1

8. There are two coins in a box. One of the coins is a fair coin and for the other, the probability of head is $\frac{2}{3}$. A coin is chosen at random from the box and tossed 5 times. Given that at least 4 heads were obtained, what is the conditional probability that the fair coin was chosen?

- (A) $\frac{729}{2521}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{243}{275}$

9. Let m be a positive integer such that $2^m + 1$ is a prime. Then which of the following statements is true?

- (A) m must be a prime
(B) m must be an odd integer
(C) m must be a power of 2
(D) m must be a power of 3

10. Let $f : [0, 1] \rightarrow (0, \infty)$ be a continuous function. Define

$$g(x) = \int_0^x f(t) dt \quad \text{for } 0 \leq x \leq 1.$$

If $g(1) = 1$, then the equation $g(x) = \frac{1}{2}$

- (A) may not have a solution
(B) may have more than one solution
(C) will have at least two solutions
(D) will have a unique solution

15. Suppose that A is an $n \times n$ matrix where $n \geq 2$. Then $A^n = 0$ if and only if

- (A) $A = 0$
- (B) all the eigenvalues of A are zero
- (C) A is singular
- (D) $\text{Rank } A \leq 1$

16. For fixed $a_1, a_2, \dots, a_5 \in \mathbb{R}$, the graph of

$$a_1x^2 + a_2x + a_3y^2 + a_4y + a_5 = 0$$

on the xy -plane can never be

- (A) a circle
- (B) an ellipse
- (C) a parabola with axis parallel to either the x -axis or the y -axis
- (D) a parabola with axis parallel to neither the x -axis nor the y -axis

17. If

$$2021^{2024} = 100m + n,$$

for some $m \in \{0, 1, 2, \dots\}$ and $n \in \{0, 1, \dots, 99\}$, then the value of n is

- (A) 21 (B) 31 (C) 61 (D) 81

18. Let A be an $m \times n$ matrix with each entry equal to $+1$, -1 or 0 such that every column has exactly one $+1$ and exactly one -1 . Then

- (A) Rank $A \leq m - 1$ (B) Rank $A = n$
(C) $m \leq n$ (D) $m - 1 \leq n$

19. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded differentiable function such that

$$f(x) + f'(x) = \sin x \quad \text{for all } x \in \mathbb{R},$$

where f' is the derivative of f . Then the value of $f(0)$ is

- (A) 0 (B) 1 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

20. Let $f(\theta) = 2(1 - \cos \theta) - \theta \sin \theta = 4 \sin \frac{\theta}{2} (\sin \frac{\theta}{2} - \frac{\theta}{2} \cos \frac{\theta}{2})$. Then which of the following assertions is true?

- (A) $f(\theta) < 0$ for $\theta \in (0, \pi)$
(B) $f(\theta) < 0$ for $\theta \in (\pi, 2\pi)$
(C) $f(\theta) > 0$ for $\theta \in (0, 2\pi)$
(D) $f(\theta)$ takes both negative and positive values in $(0, 2\pi)$

21. The value of the infinite sum

$$\sum_{n=0}^{\infty} 2^{-n} \cos \left(\frac{n\pi}{3} \right)$$

equals

- (A) 1 (B) $4 + 2\sqrt{3}$ (C) $2 + \sqrt{3}$ (D) $2 - \sqrt{3}$

26. For $x \in \mathbb{R}$, let $[x]$ denote the greatest integer less than or equal to x . Let $e^2 < a < e^3$. Then

$$\int_{1/a}^1 [\log_e x] \frac{dx}{x^2}$$

is equal to

- (A) $e^2 + e + 1 - a$ (B) $e^2 + e + 1 - 2a$
(C) $e^2 + e + 1 - 3a$ (D) $e^2 + e + 1 - 4a$

27. Let $n \geq 2$. Suppose $\alpha_1, \alpha_2, \dots, \alpha_n$ are real numbers such that $\alpha_1 < \alpha_2 < \dots < \alpha_n$. Let

$$P(x) = (x - \alpha_1)^2(x - \alpha_2) \dots (x - \alpha_n), \quad x \in \mathbb{R}.$$

If P' is the derivative of P , then $P'(x)$ has

- (A) no real root
(B) n real roots, the smallest of which is α_1
(C) $n - 1$ real roots, the smallest of which is α_1
(D) n real roots, the smallest of which is strictly larger than α_1
28. Let A be a real matrix of order 3 such that $AA^T = I_3$. Here A^T denotes the transpose of the matrix A and I_3 is the identity matrix of order 3. Assume that $\det A = -1$ and $A^T \neq A$. Let $\text{Tr } A$ denote the trace of the matrix A . Then the quantity $(\text{Tr } A)^2 - \text{Tr } A^2$ simplifies to

- (A) $\text{Tr } A$ (B) $2(\text{Tr } A)$
(C) $3(\text{Tr } A)$ (D) $4(\text{Tr } A)$

29. The 26 letters of the English alphabet are randomly permuted. What is the probability that at least two of the three letters I, N, D come next to each other?

- (A) $\frac{72}{325}$ (B) $\frac{3}{26}$ (C) $\frac{3}{13}$ (D) $\frac{3}{25 \times 26}$

30. The value of

$$\int_0^1 \int_{x^2}^1 e^{y^{3/2}} dy dx$$

is

- (A) $\frac{2}{3}(e - 1)$ (B) $e - 1$ (C) $\frac{2e}{3}$ (D) $\frac{2e}{3} - \frac{3}{2}$