1. Consider a square with four vertices A, B, C, and D arranged clockwise. A particle starts moving from the vertex A, and at any stage, it moves to either its clockwise neighbour or its anticlockwise neighbour with respective probabilities 0.6 and 0.4. Let $X_n \in \{A, B, C, D\}$ be the position of the particle after the n-th step, for $n \geq 1$.

   (a) Find $\mathbb{P}(X_{12} = A)$.

   (b) Find the stationary distribution of the Markov chain $\{X_n : n \geq 1\}$. Does the distribution of X_n converge to this stationary distribution as $n \to \infty$? Justify your answer.

   $[7 + (5 + 3) = 15]$

2. Let $X_1, X_2, \ldots, X_n$ $(n > 1)$ be independent $U(0, \theta)$ random variables where θ is an unknown positive integer.

   (a) Find the maximum likelihood estimator $\widehat{\theta}_n$ of θ.

   (b) Show that $\mathbb{P}(\widehat{\theta}_n = \theta)$ converges to 1 as $n \to \infty$.

   (c) Find the limiting distribution of $n(\theta - \widehat{\theta}_n)$.

   $[6 + 5 + 4 = 15]$

3. Let $X_1, X_2, \ldots, X_n$ and U be independent random variables, where $X_i \sim N(i, \sigma_i^2)$ and $\mathbb{P}(U = i) = \lambda_i$ for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^n \lambda_i = 1$. Define $Y = X_i$ if $U = i$. Let $V_1 = Var(Y)$ and $V_2 = \sum_{i=1}^n \lambda_i \sigma_i^2$.

   (a) Which one is bigger V_1 or V_2? Justify your answer.

   (b) Find $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $|V_1 - V_2|$ is maximum.

   (c) Find the maximum value of $|V_1 - V_2|$.

   $[5 + 7 + 3 = 15]$

4. (a) Consider the following frequency distribution.

Value (x)	1	2	3	4	5	6	7	8	9	10	Total
frequency (f)	17	35	46	38	25	13	10	9	5	2	200

Find the value of θ that minimizes $\psi(\theta) = \sum_{i=1}^{10} f_i|x_i - \theta|$, where x_i ($i = 1, 2, \ldots, 10$) denotes the i-th value of x and f_i denotes the corresponding frequency.

(b) Consider a dataset containing n bivariate observations $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. If $x_i = 2^i$ for $i = 1, 2, \ldots, n$, then show that $\Delta(\theta) = \sum_{i=1}^{n} |y_i - \theta x_i|$ is minimized at $\theta = y_n/x_n$.

[5 + 10 = 15]

5. Consider a population $\{X_1, X_2, \ldots, X_{15}\}$ of size 15, where all X_i's are distinct. A sample of size 5 is drawn from the population using simple random sampling with replacement.

(a) Show that the sample mean $\bar{x}$ can be written as $\sum_{i=1}^{15} W_i X_i$, where $(W_1, W_2, \ldots, W_{15})$ follows a multinomial distribution. Hence, derive the expectation and the variance of the sample mean when $\sum_{i=1}^{15} X_i = 30$ and $\sum_{i=1}^{15} X_i^2 = 75$.

(b) If $\widetilde{x}$ and $\widetilde{X}$ denote the sample median and the population median, respectively, then find $\mathbb{P}(\widetilde{x} = \widetilde{X})$.

[(3 + 2 + 3) + 7 = 15]

6. A researcher wants to study the effect of three different diets on the body fat percentage. For this study, the researcher gathers some volunteers from different age groups and assigns different diets to them. The allocation is shown below.

Age group	Diet 1 (Keto)	Diet 2 (Low-carb)	Diet 3 (Plant-based)	Total
Group 1 (≤ 25)	8	15	2	25
Group 2 ($26 - 40$)	0	13	7	20
Group 3 (> 40)	0	0	5	5
Total	8	28	14	50

After 3 months of following the diet, the researcher notes the change in body fat percentage (y) of the volunteers and uses the model $y_{ijk} = \mu + \tau_i + \delta_j + \varepsilon_{ijk}$, where y_{ijk} is the change in body fat percentage of the k-th individual with Diet i and in Age-group j, and ε_{ijk}'s are the errors. Assume that ε_{ijk}'s are independent and identically distributed $N(0, \sigma^2)$ variables.

(a) Are all the parameters of the model estimable? Justify.

(b) Are $\tau_i - \tau_j$ estimable for all $i \neq j$? Justify. If so, then find the best linear unbiased estimators of these elementary contrasts.

(c) Find the F-statistic for testing the equality of the diet effects on body fat percentage and find its distribution. Also, find an appropriate critical region for the test of level $\alpha \in (0, 1)$.

[$3 + 6 + 6 = 15$]

7. Let $X_1, X_2, \ldots, X_n$ be independent observations from $U(0, \theta)$ with $\theta > 0$ and $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$. Let r be a known positive integer.

   (a) Find an unbiased estimator of θ^r based on a single observation.

   (b) Find an unbiased estimator of θ^r based on a statistic that is complete and sufficient for θ.

   (c) Show that

$$\mathbb{E}[X_1^r \mid X_{(n)}] = \frac{n+r}{n(r+1)} \, X_{(n)}^r.$$

[4 + 6 + 5 = 15]

8. Consider M independent observations from a continuous univariate distribution F. Each of these observations is randomly labeled as A_1, A_2 or A_3 with probabilities p_1, p_2 and p_3, respectively, where $p_1, p_2, p_3 > 0$ and $p_1 + p_2 + p_3 = 1$. All the observations are arranged in increasing order and let R_i denote the sum of the ranks of the observations labeled A_i $(i = 1, 2, 3)$.

   (a) Find the limiting value of R_1/R_2 as $M \to \infty$.

   (b) Find the correlation coefficient between R_1 and R_2.

[8 + 7 = 15]