1. A procedure with unknown success probability p is to be tried on n patients. For each patient the procedure is conducted once. If it is unsuccessful in the first attempt, then it is conducted on the same patient for a second time. Assume that the outcome of the second trial is independent of the first. We record n_1, n_2, n_3 as the number of patients on whom the procedure was successful in the first trial, successful in the second trial, and unsuccessful in both the trials, respectively. Find the form of the most powerful test for testing $H_0 : p = \frac{1}{2}$ vs $H_a : p = \frac{2}{3}$ at a given level α. Express the test statistic in the form $a^{n_1}b^{n_2}$, for some fixed constants a and b. [15]

2. Let U_i, $i = 1, 2, \ldots; n$; be independent and identically distributed Uniform (0,1) random variables. Define $X_i = \frac{U_i}{\sum_{i=1}^{n} U_i}$.

   (a) Show that $\text{corr}(X_i, X_j) = \rho$, for all $i \neq j$, for some $\rho < 0$.

   (b) Hence show that $\rho \to 0$, as $n \to \infty$. [10 + 5 = 15]

3. In a game, a player starts with an initial score 0. The player rolls two fair dice, and the difference between two outcomes (say, X_1 and X_2) is noted. If the difference $D = |X_1 - X_2|$ is positive, it is added to the score of the player, and the process continues. If the difference is zero the process stops, and the player's current score is considered as the final score S. Find the expected value of S, and show that the median of S cannot exceed 24. [9 + 6 = 15]

4. Let $Y_1 \sim$ Poisson (λ), and $Y_2 = \sum_{i=1}^{Y_1} B_i + \epsilon_2$, where B_i's are independent and identically distributed Bernoulli (ρ) random variables, and $\epsilon_2 \sim$ Poisson$((1-\rho)\lambda)$, independently of Y_1. Also, B_i and ϵ_2 are independent, for all i. Show that

   (a) $Y_2 \sim$ Poisson(λ),

   (b) $\text{corr}(Y_1, Y_2) = \rho$. [7 + 8 = 15]

5. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed Poisson (λ) random variables, and $\bar{X}$ be the sample mean. Define $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$.

(a) Show that $\bar{X}$ is the Uniformly Minimum Variance Unbiased Estimator of λ.

(b) Show that $E(S^2|\bar{X}) = \bar{X}$.

(c) Hence show that $\text{Var}(S^2) > \text{Var}(\bar{X})$. [5 + 7 + 3 = 15]

6. There are 5 plots in each of 4 blocks, and the plots within each block are homogeneous. Four treatments A, B, C and D are allotted randomly in each block such that A is applied twice in each block whereas each of B, C, D are applied only once in each block. Let $\tau_1, \tau_2, \tau_3, \tau_4$ be the treatment effects of A, B, C, D, respectively. Assume that a response is measured from each plot, and the errors are independent and identically distributed $N(0, \sigma^2)$ random variables. Find the least squares estimators of $\tau_1 - \tau_j$, for $j = 2, 3, 4$; and also find the variances of these estimators. [9 + 6 = 15]

7. Let $\{X_n\}_{n\geq 1}$ and X be integrable random variables defined on the same probability space $(\Omega, \mathscr{A}, P)$. Show that $E(|X_n - X|) \to 0$, as $n \to \infty$, if and only if $\int_A X_n dP \to \int_A X dP$ uniformly in $A \in \mathscr{A}$. [15]

8. Consider a function $f : [0, 1] \to \mathbb{R}$, whose form is given upto unknown parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}$, by

$$f(x) = \begin{cases} \alpha_1 + \beta_1 x + \gamma_1 x^2, & \text{if } x \leq 0.5, \\ \alpha_2 + \beta_2 x + \gamma_2 x^2, & \text{if } x > 0.5. \end{cases}$$

Let $x_i = \frac{i}{n}$, for $i = 0, 1, 2, \ldots, n$, and suppose that we have collected independent observations $Y_i \sim N(f(x_i), \sigma^2)$. We are interested in estimating the parameters that determine f using this data.

(a) Suppose it is given that $\gamma_1 = \gamma_2 = 0$. Obtain the maximum likelihood estimators of the remaining parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ and σ^2.

(b) Suppose now that γ_1, γ_2 are also unknown, and we wish to obtain the maximum likelihood estimators of $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ and σ^2. Explain how you will proceed, indicating the key steps only.

[10 + 5 = 15]