

1. Consider a Markov chain $\{X_n : n \geq 0\}$ with state space $S = \{0, 1, 2, 3\}$ and stationary transition probability matrix $P = ((p_{ij}))$ satisfying

$$\sum_{j=0}^3 j p_{ij} = i \quad \text{for } i = 0, 1, 2, 3.$$

Show that 0 and 3 must be absorbing states. [15]

2. Suppose that (X_i, Y_i) , $i \geq 1$, are independent and identically distributed bivariate random vectors with $E(X_1) = \mu_x$, $E(Y_1) = \mu_y$, $Var(X_1) = \sigma_x^2$, $Var(Y_1) = \sigma_y^2$ and $Corr(X_1, Y_1) = \rho$. If X_1 and Y_1 are positive random variables, show that

$$Z_n = \sqrt{n} \left(\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n Y_i} - \frac{\mu_x}{\mu_y} \right)$$

converges in distribution to a normal random variable with mean 0 and variance $\frac{1}{\mu_y^4}(\mu_y^2 \sigma_x^2 + \mu_x^2 \sigma_y^2 - 2\rho \mu_x \mu_y \sigma_x \sigma_y)$. [15]

3. Consider a sequence of random variables $\{X_n : n \geq 1\}$, where $X_n \sim N(0, n^{-\alpha})$ for $n = 1, 2, \dots$ and $\alpha > 0$ is fixed. Show that X_n converges almost surely to 0. [15]

4. Let X_1, X_2, \dots, X_n be independent Bernoulli random variables with $P(X_i = 1) = p$ for $i = 1, 2, \dots, n$, where $p \in [1/2, 1)$.

(a) Is $T = n^{-1} \sum_{i=1}^n X_i$ a minimum variance unbiased estimator of p ? Justify your answer.

(b) Find an estimator T_0 such that $E(T - p)^2 > E(T_0 - p)^2$ for all $p \in [1/2, 1)$. [7+8]

5. Let X_1, X_2, \dots, X_n be independent and identically distributed normal variables with mean θ and variance 1. For the testing problem $H_0 : \theta = 0$ against $H_1 : \theta = 1$, find the critical region that minimizes $P(\text{Type I Error}) + 3 P(\text{Type II Error})$. [15]

6. Consider the following data set

x	1	2	3	4	5	6
y	18.6	15.0	10.6	7.2	3.4	-1.1

Find the equation of the line $y = \alpha + \beta x$ that passes through the point $(5, 3)$ and minimizes $\sum_{i=1}^6 |y_i - \alpha - \beta x_i|$, where (x_i, y_i) denotes the i -th ($i = 1, 2, \dots, 6$) observation. [15]

7. Consider the following block design D with 3 treatments denoted by 1, 2 and 3, assigned in 3 blocks of size 5.

$$D = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 2 & 3 \\ 1 & 2 & 3 & 3 & 1 \end{bmatrix}$$

- (a) Identify a set of maximum number of linearly independent estimable functions of the treatment effects under the usual additive fixed effects two way ANOVA model.
- (b) Find the best linear unbiased estimator of $\tau_1 - \tau_2$, where τ_i denotes the i -th ($i = 1, 2, 3$) treatment effect. [3+12]
8. Let θ be the unknown proportion of tax evaders (denoted by attribute A) in a finite population of size N . In order to estimate θ , a statistician chose a sample of size n using simple random sampling without replacement. Because of the sensitive nature of A, a fair die was used. Two faces of the die were inscribed with A and the other four faces were inscribed with A^c . Each person in the sample was asked to roll the die till s/he got the face showing her/his true state and only report Z , the number of rolls required for it. These responses were denoted by Z_1, Z_2, \dots, Z_n . Find
- (a) an unbiased estimator T for θ ,
- (b) an unbiased estimator for $\text{Var}(T)$. [9+6]