1. Define $x_1 = 1$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ for all $n = 1, 2, \ldots$.

   (a) Verify whether the sequence $\{x_n\}$ is convergent.

   (b) Find $\liminf_{n\to\infty} x_n$.

   [9 + 3 = 12]

2. Suppose $f : \mathbb{R}^d \mapsto \mathbb{R}$ is a continuous function with a unique maxima at some $\mathbf{x}_0 \in \mathbb{R}^d$. Then, show that for any $\delta > 0$,

$$f(\mathbf{x}_0) > \sup\left\{f(\mathbf{x}) : \|\mathbf{x} - \mathbf{x}_0\| \geq \delta\right\}.$$

   Here, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$.

   [12]

3. Suppose $\mathbf{u} \in \mathbb{R}^d$ is a unit vector and $\mathbf{I}_d$ is the $d \times d$ identity matrix. Define $\mathbf{A} = \mathbf{I}_d - \mathbf{u}\mathbf{u}^\top$. Here, $\mathbf{u}^\top$ denotes the transpose of $\mathbf{u}$.

   (a) Verify whether $\mathbf{A}$ is a positive semi-definite matrix.

   (b) Find the rank of $\mathbf{A}$.

   [4 + 8 = 12]

4. A person is asked to paint four pillars A, B, C, and D located at the four corners of a rectangular football ground. The person is allowed to use only one colour on a pillar. However, more than one pillars can be painted with the same colour. Assume that the person has four colours: green, maroon, red and yellow, and chooses a colour at random with equal probability to paint a pillar. Given that any two pillars located at adjacent corners are painted with different colours, what is the probability that all four colours are used?

   [12]

5. Let X be a random variable taking values in the interval $[1, 10]$.

   (a) If $\mathbb{E}(X) = 2.8$, then show that the standard deviation of X can not exceed 3.6.

   (b) If X_1 and X_2 are two independent random variables having the same distribution as X, then show that $\mathbb{E}\left(\frac{X_1}{X_2}\right)$ can not be smaller than 1.

$[6 + 6 = 12]$

6. Consider three random variables X, Y, and Z where $Z \sim U(0, 1)$, and for any given value z of Z, the other two variables X and Y are independent and identically distributed as $N(z, 1)$.

   (a) Verify whether X and Y are independent.

   (b) Does (X, Y) follow a bivariate normal distribution? Justify your answer.

$[6 + 6 = 12]$

7. Suppose that $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are independent observations from a bivariate normal distribution with the density

$$f_\theta(x, y) = \frac{1}{2\pi\sqrt{1 - \theta^2}} \exp\left\{-\frac{1}{2(1 - \theta^2)}(x^2 + y^2 - 2\theta xy)\right\},$$

   where $-\infty < x, y < \infty$ and $\theta \in (-1, 1)$.

   (a) Find a minimal sufficient statistic for θ.

   (b) Define $Q_1 = \sum_{i=1}^{n} X_i^2$ and $Q_2 = \sum_{i=1}^{n} Y_i^2$. Argue whether the following statistics are ancillary for θ:
   (i) Q_1, (ii) $Q = (Q_1, Q_2)$.

$[4 + (3 + 5) = 12]$

8. Let θ_1, θ_2, and θ_3 be the true angles of a huge triangular field. Suppose that each of these angles is measured by a separate person subject to an additive error. Assume that these three errors are independently distributed as $N(0, \sigma^2)$ where σ^2 is unknown. Let these measurements be Y_1, Y_2, and Y_3, respectively. Construct a test of level α $(0 < \alpha < 1)$ for testing $H_0 : \theta_1 = \theta_2$ against $H_1 : \theta_1 \neq \theta_2$.

[12]

9. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables with the probability density function

$$f_\theta(x) = 2\, e^{-2(x-\theta)}\ \mathbb{I}\{x \geq \theta\},$$

where $\theta \in \mathbb{R}$ and $\mathbb{I}\{\cdot\}$ is the indicator function. Let θ_0 and θ_1 be two known real numbers with $\theta_1 > \theta_0$. For a prefixed $\alpha \in (0, 1)$, derive a randomized most powerful test of size α for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ when

(a) $e^{2n(\theta_0 - \theta_1)} < \alpha$,

(b) $e^{2n(\theta_0 - \theta_1)} > \alpha$.

[6 + 6 = 12]

10. Suppose (X_1, X_2, X_3, X_4) follows a multivariate normal distribution with the mean $(0, 0, 0, 0)$ and the dispersion matrix

$$\begin{pmatrix} 25 & 0 & 0 & 1 \\ 0 & 16 & 4 & 0 \\ 0 & 4 & 9 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}.$$

(a) Find $\mathbb{P}(X_1 > X_2 X_3 X_4)$.

(b) Compute the multiple correlation coefficient of X_1 with (X_2, X_3, X_4).

(c) Compute the partial correlation coefficient between X_2 and X_3 given (X_1, X_4).

[6 + 3 + 3 = 12]