1. Let $g : [0, 1] \to \mathbb{R}$ be a continuous function such that $g(1) = 0$. Show that $\sup_{x \in [0,1]} |x^n g(x)| \to 0$ as $n \to \infty$.

[12]

2. Let $f : [0, \infty) \to [0, \infty)$ be an increasing function. For all $t > 0$, define $g(t) = \left(\int_0^t f(x)dx \right) / t$. Check whether g is an increasing function.

[12]

3. Suppose $\mathbf{A}$ and $\mathbf{B}$ are two $n \times n$ matrices with real entries such that the sum of their ranks is strictly less than n. Show that there exists a nonzero column vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{Bx} = \mathbf{0}$.

[12]

4. In a tennis match between twin brothers Bikas and Bimal, Bikas wins a set with a probability of 0.6. A best-of-five tennis match stops as soon as a player wins three sets. Assume that the outcomes of the sets are independent.

   (a) If the match lasts for five sets, what is the probability that Bikas will win the match?

   (b) If Bimal wins the match, what is the probability that the match lasts for five sets?

[4+8 = 12]

5. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed discrete uniform random variables taking values $1, 2, 3, \ldots, \theta$, where θ is an unknown *positive integer*.

   (a) Define $\Psi(x) = xe^x - (x-1)e^{x-1}$ for all $x \in \mathbb{R}$. Find $\mathbb{E}(\Psi(X_1))$.

   (b) Find the uniformly minimum variance unbiased estimator (UMVUE) of e^θ.

[4+8 =12]

6. If $X_1, X_2, \ldots, X_n$ are independent and identically distributed $U(0, \theta)$ random variables, show that their geometric mean is a consistent estimator of θ/e.

[12]

7. A bivariate random vector (X, Y) follows uniform distribution on the square region with four vertices $(\theta, 0), (0, \theta), (-\theta, 0)$ and $(0, -\theta)$, where $\theta > 0$. Suppose that $(X_1, Y_1), \ldots, (X_n, Y_n)$ are n independent copies of (X, Y).

   (a) Find a real-valued sufficient statistic for θ.

   (b) Find the maximum likelihood estimator of θ.

[6+6=12]

8. Consider a population $\{X_1, X_2, \ldots, X_{25}\}$ consisting of 25 observations, where each X_i $(i = 1, 2, \ldots, 25)$ takes integer values between 0 and 9 (both inclusive). Let the population mean be 5.4. Consider a simple random sample $\{x_1, x_2, \ldots, x_5\}$ of size 5 drawn without replacement from this population. Define $\bar{x} = (x_1 + \cdots + x_5)/5$.

   (a) Show that $\mathrm{V}ar(\bar{x}) = \sigma^2/6$, where σ^2 is the population variance.

   (b) Show that $\mathrm{V}ar(\bar{x})$ lies between 0.04 and 3.24.

[3+9=12]

9. Construct a test for $H_0 : \mu = 0$ against $H_1 : \mu = 1$ based on a single observation from $N(\mu, 1)$ such that

$$\mathbb{P}(\text{Type I error}) + 2\,\mathbb{P}(\text{Type II error})$$

is minimized. Justify your answer.

[12]

10. Consider a linear regression model $Y_i = \beta_1 X_{1i} + \beta_2 X_{2i} + \epsilon_i$ $(i = 1, 2, \ldots, n)$ with two non-stochastic covariates X_1 and X_2, where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent and identically distributed $N(0, \sigma^2)$ variates. Assume that

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} X_{1i} = \sum_{i=1}^{n} X_{2i} = 0, \quad \sum_{i=1}^{n} X_{1i}^2 = \sum_{i=1}^{n} X_{2i}^2 = 1$$

$$\text{and } \sum_{i=1}^{n} X_{1i} X_{2i} = \alpha \text{ for some } \alpha \in (-1, 1).$$

Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be the least squares estimates of β_1 and β_2, respectively, such that $\hat{\beta}_1 > \hat{\beta}_2 > 0$.

(a) Show that the sample correlation coefficient between Y and X_1 is larger than that between Y and X_2.

(b) For any $(\ell_1, \ell_2) \in \mathbb{R}^2$ with $\ell_1^2 + \ell_2^2 = 1$, show that the variance of $\ell_1 \hat{\beta}_1 + \ell_2 \hat{\beta}_2$ cannot be smaller than $\sigma^2/(1 + |\alpha|)$.

[6+6 =12]