1. Consider $a_n > 0$, for $n = 1, 2, \ldots$, such that $\sum_{n=1}^{\infty} a_n = \infty$. Check whether $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$ is convergent if $\liminf na_n > 0$. [10]

2. Let X_1 and X_2 be independent and identically distributed random variables with probability density function (pdf) $f(x) = nx^{n-1}$, for $0 < x < 1$, for fixed n ($n > 1$). Show that $P(X_1 < X_2 | X_1 < nX_2) = \frac{1}{2-\frac{1}{n^n}}$. [10]

3. Define the matrix P as follows:

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ c & d & 0 & 0 \end{bmatrix}.$$

(a) For what values of (a, b, c, d), P is the transition probability matrix of a Markov Chain with four states?

(b) For different values of (a, b, c, d) find all disjoint, closed, irreducible, recurrent classes. $[4 + 6 = 10]$

4. A particle located at a vertex of a square ABCD moves to one of the neighbouring vertices with equal probability. Suppose that the particle starts moving from the vertex A. Find the probability that it will visit each of the other three vertices at least once before returning to A. [10]

5. Find the dimension of the vector space of all 4×4 matrices whose row sums and column sums are all equal. Justify your answer. [10]

6. Suppose X follows a Poisson distribution with variance θ, and Y is a random variable which equals to X with probability 0.5, and equals to 0 with probability 0.5.

(a) Find the coefficient of variation of Y.

(b) Show that Variance $(Y) > E(Y)$.

(c) How would you estimate θ if it is known that 64 out of 100 observations for Y are 0? $[5 + 2 + 3 = 10]$

7. Consider a function $f : (0,1) \to \mathbb{R}$ which is differentiable on $(0,1)$. It is also known that $|f'(x)| < M$, for all $x \in (0,1)$, for some $M > 0$. Define $a_n = f(\frac{1}{n+1})$, for $n \geq 1$. Does $\lim_{n\to\infty} a_n$ exist? Justify your answer. [10]

8. Consider a sequence of n observations where each observation is an independent realization of a Uniform $\{0,1,2,\ldots,9\}$ random variable. We define a "run" as a subsequence of the same integer which is preceded and followed by a different integer or no integer at all. Let R_n be the number of runs in a sequence of n observations. Find $E(R_n)$. [10]

9. Two points are chosen independently from Uniform(0,1) distribution to divide a line of unit length into three smaller line segments.
(a) Find the probability that a triangle can be formed using the segments.
(b) If they form a triangle, find the probability that the area of the triangle will be bigger than $\frac{1}{8}$ square unit. [7 + 3 = 10]

10. Let θ_i be independent and identically distributed Uniform$(0, 2\pi)$, and define $Y_i = |\sin\theta_i|$. Find the limit to which $\frac{1}{n}\sum_{i=1}^{n} Y_i$ converges in probability. [10]

11. Suppose we want to estimate the probability that a randomly selected student smokes. Since smoking among students is a sensitive issue, a student may not answer correctly when asked whether s/he smokes. For this we adopt a different method. We construct 40 cards and write 'I smoke' on 25 cards and 'I do not smoke' on rest of the cards. A sample of 300 students are selected randomly. Now we ask each student in the sample to select randomly one of the 40 cards and respond 'yes' or 'no' without disclosing the question. Here there is no way that the experimenter knows what is written on the card chosen by each student. Selection of card is made with replacement. Let A be the event that a student answers 'yes' and S denote that a randomly selected student smokes.

(a) Establish a relationship between $P(A)$ and $P(S)$.

(b) If 130 students answered 'yes' out of 300 students, find an estimate of $P(S)$ using the result in part (a). Check whether this estimator is unbiased.

(c) Find the variance of the estimator of $P(S)$. [$3 + 4 + 3 = 10$]

12. A novice game shooter successfully hits the target with probability θ ($\theta > 0.5$) when the shooter shoots from 10 meters distance. If the distance is increased to 25 meters, the probability reduces to $1 - \theta$. During a practice session the shooter is asked to hit the target from 10 meters and 25 meters distances, and the number of successful hits is counted. This process is repeated 100 times, and the results are given below. Based on the information given in the table, find the maximum likelihood estimate of θ.

No. of successful hits	0	1	2	Total
Frequency	23	58	19	100

[10]