

1. Consider a sequence $\{a_n; n \geq 1\}$ of real numbers, where

$$a_{n+1} = \frac{3}{2}a_n - \frac{1}{2}a_{n-1} \text{ for all } n > 1.$$

- (a) Show that the sequence converges.
 (b) Also find the limiting value of the sequence in terms of a_1 and a_2 . [8+4]

2. Let f be a real valued function defined on $[0, \infty)$ such that f is continuous on $[0, \infty)$, $f(0) = 0$ and f' is non-decreasing on $(0, \infty)$. Define $g(x) = f(x)/x$ for all $x \in (0, \infty)$. Show that g is non-decreasing on $(0, \infty)$. [12]

3. Let $\mathbf{A}_{m \times m} = \begin{pmatrix} \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} \\ & \mathbf{P}_{m-1 \times m} & & \end{pmatrix}$ be an orthogonal matrix and \mathbf{B} be an $m \times m$ symmetric matrix with rank $m - 1$ and $\mathbf{B}\mathbf{1}_m = \mathbf{0}$, where $\mathbf{1}_m = (1, 1, \dots, 1)^T$ denotes the m -dimensional vector with all elements equal to 1. Show that

- (a) $\mathbf{P}^T\mathbf{P} = \mathbf{I} - \frac{1}{m}\mathbf{1}_m\mathbf{1}_m^T$, where \mathbf{I} is the $m \times m$ identity matrix,
 (b) rank of \mathbf{PBP}^T is $m - 1$.

Note: For a matrix \mathbf{M} , its transpose is denoted by \mathbf{M}^T . [4+8]

4. A fair coin is tossed repeatedly and let \mathcal{T} be the number of tosses till two consecutive tails are observed for the first time.

- (a) Show that

$$E(\mathcal{T} \mid \text{tail is observed in the first toss}) = 2 + \frac{1}{2}E(\mathcal{T}).$$

- (b) Find a similar formula for

$$E(\mathcal{T} \mid \text{head is observed in the first toss}).$$

- (c) Compute $E(\mathcal{T})$. [6+3+3]

5. Consider a population consisting of k classes with proportions p_1, p_2, \dots, p_k , where $p_i \in (0, 1)$ for every $i = 1, 2, \dots, k$ and $p_1 + p_2 + \dots + p_k = 1$. Let N denote the number of classes not represented in a random sample of size n drawn with replacement from the population. Find $E(N^2)$. [12]

6. Let U and V be two dependent discrete random variables, each being uniformly distributed on $\{1, 2, \dots, k\}$. Let W be another random variable having the same uniform distribution but independent of U and V . Define a random variable $X = (V + W) \bmod k$. Show that

(a) X is uniformly distributed on $\{0, 1, 2, \dots, k - 1\}$,

(b) U and X are independent. [6+6]

7. Consider a data set $(x_1, y_1), (x_2, y_2), \dots, (x_{100}, y_{100})$, where $x_i = a$ for all $i \leq 50$ and $x_i = b$ for all $i > 50$ ($a \neq b$). Two regression functions

$$y = \alpha_0 + \alpha_1 x \quad \text{and} \quad y = \beta_0 + \beta_1 x^3$$

are fitted to this data set using the method of least squares. Which of these two models will lead to smaller residual sum of squares? Justify your answer. [12]

8. Let X_1, X_2, \dots, X_n be independent and identically distributed normal random variables with mean θ and variance 1, where $\theta \geq 0$. Find

(a) the maximum likelihood estimator $\hat{\theta}_n$ of θ ,

(b) the asymptotic distribution of $T_n = \sqrt{n}\hat{\theta}_n$ when $\theta = 0$. [4+8]

9. Let X_1, X_2, \dots, X_n be a random sample from a continuous distribution. For each $j = 1, 2, \dots, n$, let

$$R_j = \#\{i : X_i \leq X_j, 1 \leq i \leq n\}.$$

Thus, R_j is the number of random variables X_i ($i = 1, 2, \dots, n$) which are less than or equal to X_j .

(a) Find the correlation coefficient between R_1 and R_n .

(b) For any fixed k ($1 < k < n$), find the correlation coefficient between $Y_1 = \sum_{j=1}^k R_j$ and $Y_2 = \sum_{j=k+1}^n R_j$. [9+3]

10. For $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, define $\|\mathbf{x}\| = \sqrt{x_1^2 + \dots + x_d^2}$.

(a) Show that $f(\mathbf{x}) = \|\mathbf{x}\|$ is a convex function.

(b) Let \mathbf{X} be a d -dimensional random vector symmetrically distributed about the origin (i.e. \mathbf{X} and $-\mathbf{X}$ have the same distribution). Show that $\psi(\boldsymbol{\theta}) = E\|\mathbf{X} - \boldsymbol{\theta}\|$ is minimized at $\boldsymbol{\theta} = \mathbf{0}$. [4+8]