**Group A**

# *Mathematics*

1. (a) Suppose $A$ is a $3 \times 3$ matrix as follows

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Show that $A^n - A^{n-2} = A^2 - I$, for all integers $n \geq 3$.

(b) Formulate the given limit as an integral and find its value

$$\lim_{n \to \infty} \left\{ \left( \frac{1}{n+1} \right) + \left( \frac{1}{n+2} \right) + \cdots + \left( \frac{1}{n+n} \right) \right\}.$$

(c) Let $V$ be a vector space of twice differentiable real-valued functions on $\mathcal{R}$ satisfying $f'' - 2f' + f = 0$. Define $T : V \to \mathcal{R}^2$ by $T(f) = (f'(0), f(0))$. Show that $T$ is both one-to-one and onto.

(d) Consider a $3 \times 3$ matrix $A$, where

$$A = \begin{pmatrix} 1 & 1+x & 1+x+x^2 \\ 1 & 1+y & 1+y+y^2 \\ 1 & 1+z & 1+z+z^2 \end{pmatrix}, \quad x, y, z \in \mathcal{R}.$$

Show that $det(A) = (z - y)(z - x)(y - x)$.

$$[6 + 5 + 8 + 5 = 24]$$

# *Probability & Statistics*

2. (a) Suppose $X$ and $Y$ are two independent and normally distributed random variables with common mean $\mu$ and variance $\sigma^2$. If $P(X + Y > 27) = P(X - Y < 2)$, then calculate the mean $\mu$.

(b) i. Explain the concept of multicollinearity in multiple linear regression model.

ii. The estimate of coefficients using Ordinary Least Squares method becomes imprecise in the presence of multicollinearity - discuss with an example.

(c) Let us choose a point uniformly from a disk of radius $R$ $(> 0)$. Let the points on the plane be determined by their Cartesian coordinates $(x, y)$. Let $X$ and $Y$ be random variables denoting the co-ordinates of points chosen. Find $E(X)$ and $V(X)$, where $E(X)$ and $V(X)$ are expected value and variance of $X$ respectively.

(d) While planning for the requirement of electricity in a city, the authorities assumed that the consumption per household to be normally distributed with a mean of 150 units. Since some of the officials were sceptical that the mean consumption unit was substantially more, a study was conducted to verify it. A sample of 100 selected households showed a mean of electricity consumption at 155 units with a standard deviation of 20 units. Based on the above information, do you think that the assumption of the authorities was correct at 5% level of significance?

Consider $t_{0.05,99} = 1.66$, $t_{0.025,99} = 1.98$, $Z_{0.95} = 1.64$, $Z_{0.975} = 1.96$, $\chi^2_{0.05,99} = 123.22$, $\chi^2_{0.025,99} = 128.42$ as the upper tail values of the corresponding distributions.

$$[6 + (2 + 4) + 8 + 4 = 24]$$

3. (a) Let $X$ and $Y$ be two continuous random variables with joint probability density function

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1, \ 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Compute $P\left(|X - Y| \leq \frac{1}{2} \mid X + Y \leq 1\right)$.

(b) Let $X$ be an observation from the probability mass function

$$f(x) = \left(\frac{\theta}{2}\right)^{|x|} (1 - \theta)^{1-|x|}, \quad x \in \{-1, 0, 1\}; \quad 0 \leq \theta \leq 1.$$

   i. Find the maximum likelihood estimator $\hat{\theta}(X)$ of $\theta$.

   ii. Define another estimator

$$T(X) = \begin{cases} 2, & \text{if } x = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\hat{\theta}(X)$ is more precise in unbiasedly estimating $\theta$ than $T(X)$.

(c) Let $\{Z_n, n = 0, 1, 2, \ldots\}$ be a Markov Chain with state space $\{0, 1, 2\}$ and the transition probability matrix $P$ is given as

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & a & 1-a \\ b & 0 & 1-b \end{bmatrix}$$

   i. Determine the conditions on $a$ and $b$ which ensure that the Markov Chain is irreducible.

   ii. Assume $a = b = \frac{1}{3}$. Find the stationary distribution of the Markov chain, if it exists.

$$[8 + (4 + 4) + (5 + 3) = 24]$$

## Group B

## *Operations Research*

4. (a) Let $M$ be an $n \times n$ positive definite and diagonal matrix. Define the $M$-norm of a vector $x$ such that $||x||_M = \sqrt{x^T M x}$. Find a necessary and sufficient condition for $x^*$ to be a global minimizer of the following problem.

$$min_{x \in \mathcal{R}^n} h(x) = x^T g + \tfrac{1}{2} x^T H x, \text{ subject to } ||x||_M \leq \delta,$$

where $\delta > 0$, $g \in \mathcal{R}^n$, $H$ is an $n \times n$ real symmetric matrix and $a^T b$ represents the Euclidean inner product of two vectors $a, b \in \mathcal{R}^n$.

(b) Let $x^*$ be a basic feasible solution of the linear programming problem:

$$max \ \ f = \beta^T x \text{ subject to } Ax = b, x \geq 0,$$

where $A$ is an $m \times n$ matrix, $b \in \mathcal{R}^m$, $\beta, x \in \mathcal{R}^n$. Prove that $x^*$ is an extreme point in the set of feasible solutions.

(c) Consider a two person zero sum game for which Player I has $m$ strategies, Player II has $n$ strategies and $A = (a_{ij})$ is the pay-off matrix. Assume that this game has the solution of mixed stationary strategies. Formulate the problem in such a way to find the solution of mixed strategies with maximum number of zeros.

$$[8 + 8 + 8 = 24]$$

5. (a) The annual demand quantity for glass frames is 100000 and each glass frame costs Rs. 20. The ordering cost is Rs. 80 per order with an inventory holding cost of Rs. 25 per glass

frame every year. If the order quantity is 1000 or more, the supplier will offer a 5% discount, whereas the discount is 10% if the order is 2000 or more. If the total cost is to be minimized then which discount scheme is recommended for the retailer?

(b) Trucks arrive at the toll gate at a rate of 70 per hour with Poisson distribution. The service rate at the toll gate follows Poisson distribution with a rate of 80 trucks per hour. Truck drivers complain about long waiting time, and management is willing to add one more toll gate to reduce the mean processing time to 35 seconds. Calculate the idle time and the average queue length on adding a new toll gate.

(c) Two children, A and B, engage in a game in which they randomly select a ball from their respective bags, each containing red and green balls. If both select red balls simultaneously, child A earns Rs. 10. Should both pick green balls, child A receives Rs. 3. Conversely, if the children choose balls of different colors, child B wins and receives Rs. 6. What strategies should children adopt? If given the chance to place a bet, which child would you choose to support and why?

$$[8 + 8 + 8 = 24]$$

## Reliability

6. (a) Let $X$ be a continuous random variable with a non-negative support denoting lifetime of a manufactured product. Let the probability density function and reliability function of $X$ be given by $f(.)$ and $R(.)$, respectively. Answer the following questions:

i. Show that the variance of $X$ is given by
$2 \int_0^\infty x R(x) dx - \left( \int_0^\infty R(x) dx \right)^2$.

ii. Suppose $R(x) = \exp\{1 - \exp\{\sqrt{1.5x}\}\}, x \geq 0$. Find the time $x_0$ at which the hazard function $h(x)$ of $X$ changes from decreasing to increasing function.

iii. The mean residual life function of $X$ at some point time $x > 0$ is defined by $\phi(x) = \mathrm{E}(X - x | X > x)$. Show that $R(x) = \frac{\phi(0)}{\phi(x)} \exp\{- \int_0^x \frac{dt}{\phi(t)}\}$.

(b) Consider a system with four components. The system functions only if component 1 and at least one of the remaining three components function. If $T_i$ is the life time of the components where the hazard rate of the $i$-th component is $\lambda_i$, $i = 1, 2, 3, 4$, find the reliability and hazard rate of the system at time $t$.

$$[(6+6+7)+ 5 = 24]$$

7. (a) Consider a scenario where 6 identical units are placed on a life test. Stop the experiment if 4 failures are observed. The observed failure times in hours are given as $4, 3, 6$ and $1$.

Suppose, the failure time of a unit with probability density function is

$$f(t) = ct \exp\left( -\frac{ct^2}{2} \right), \quad t \geq 0, \ c > 0.$$

i. Find the maximum likelihood estimate (MLE) of the parameter $c$ (approximate up to 2 decimal places).

ii. Find the MLE of the reliability of an item at 8 hours, given that, it has already survived upto 6 hours.

(b) The time to failure of a system follows Weibull distribution with a given shape and scale parameters. What is the probability that the system survives beyond its mean time to failure?

(c) The hazard rate of an item is given by

$$\lambda(x) = \begin{cases} x^2, & \text{if } 0 < x \le x_0 \\ x_0(2x - x_0), & \text{if } x > x_0 \end{cases}$$

where $x_0$ is a positive constant.

   i. Derive the reliability function.

  ii. Find the mean time to failure of the item.

$$[(6+4)+6+(5+3)=24]$$

## *Statistical Quality Control*

8. (a) The process capability index $C_p$ is to be computed for a quality characteristic $X$, which follows normal distribution with unknown mean $\mu$ and unknown standard deviation $\sigma$. Find an unbiased estimator of $C_p$ and its distribution.

(b) Is it possible for a process to be in control and still produce nonconforming output? Explain some corrective measures under these circumstances.

(c) A process engineer wants to monitor the process with respect to proportion defective. The engineer selects 1000 samples per shift and expects the defective rate to be around 5%. Suggest an appropriate control chart with justification.

(d) Explain the difference between specification limits and control limits. Is there a desired relationship between these two?

(e) What are the advantages and disadvantages of EWMA charts compared to Shewhart control charts?

$$[8 + 4 + 4 + 4 + 4 = 24]$$

9. (a) In a process capability analysis, prove the relationship $C_{pk} = C_p(1 - k)$. Here, $C_p$ is potential process capability index, $C_{pk}$ is process performance index, $k = \frac{|\mu - T|}{(USL - LSL)/2}$, $T$ is the midpoint of the upper specification limit (USL) & lower specification limit (LSL), and $\mu$ is the process mean.

   (b) Suggest action plans to improve the process performance for statistically i) in-control and ii) out-of-control processes.

   (c) What is the importance of operating characteristics (OC) curve in selection of the sampling plans? Describe the impact of sample size and the acceptance number on the OC curve. What is the disadvantage of having an acceptance number of zero?

   (d) For a single sampling inspection plan $(10, 1)$ with AQL $= 1\%$ and LTPD $= 4\%$, derive the expressions for producer's risk and consumer's risk.

$$[8 + 4 + 8 + 4 = 24]$$