QRB
2019

Group A

Mathematics

1. (a) Let W be the vector space of all linear transformations from $\boldsymbol{R}^2 \to \boldsymbol{R}^2$, that maps the line $y = 2x$ to itself. What is the dimension of W?

   (b) Construct a 3×3 non-zero matrix N such that $N^3 = O$, but $N \neq O$ & $N^2 \neq O$, where O is a 3×3 null matrix. Show that each eigenvalue of N is zero.

   (c) Let $W = \{X_1, X_2, ..., X_{10}\} \in \boldsymbol{R}^{10}$ such that,

   $$X_{n+2} = X_{n+1} + X_n,\ n \geq 1.$$

   Find the dimension of the set W.

   (d) Find the set of values of x such that $\log_7 \frac{x-2}{x-3} < 0$.

   [6+6+6+6 = 24]

2. (a) The following system of equations

   $$x = (\sin\theta)y + (\cos\theta)z$$

   $$y = z + (\cos\theta)x$$

   $$z = (\sin\theta)x + y$$

   has a nontrivial solution, for $0 \leq \theta \leq \pi$. Find the value of $\frac{4\theta}{\pi}$.

   (b) Suppose n is an odd positive integer and

   $$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots + a_n = 0,$$

   where $a_0, a_1, \cdots, a_n$ are real numbers. Show that the polynomial

   $$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

   has a real root in $[0, 1]$.

(c) A cubic polynomial $f(x)$ vanishes at $x = -2$ and has local minimum and maximum at $x = -1$ and $x = \frac{1}{3}$ respectively. If

$$\int_{-1}^{1} f(x)dx = \frac{14}{3},$$

find $f(x)$.

[8+8+8=24]

Probability & Statistics

3. (a) The random variables (X, Y) have joint probability mass function

$$P(X = x, Y = y) = \frac{e^{-2}}{y!(x-y)!},$$

$x = 0, 1, \ldots, \quad y = 0, 1, \ldots, x.$

Find the conditional distribution of Y given $X = x$. Identify the distribution. Find $E(Y|X = x)$.

(b) The random variables (X_1, X_2, X_3) have the joint distribution with mean $(0, 0, 0)'$ and variance-covariance matrix

$$\Sigma = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1 & 0.55 \\ 0.8 & 0.55 & 1 \end{bmatrix}.$$

Find the correlation coefficient between X_1+X_3 and X_2-X_3.

(c) Consider an experiment involving two dice. One is a standard die with faces numbering from 1 to 6. The other die has three ones and three fours. Find the probbaility of getting two fours from the following experiment:"Toss a fair

coin; roll the standard die four times if head appears and the other die four times if tail appears".

[(6+2+2) + 6 + 8 =24]

4. (a) Suppose that $X_{11}, \ldots, X_{1n_1}$ and $X_{21}, \ldots, X_{2n_2}$ are independent random samples from normal distributions with respective unknown means μ_1 and μ_2 and variances σ^2 and $2\sigma^2$.

   (i) Find the maximum likelihood estimators of μ_1, μ_2 and σ^2.

   (ii) Derive a $100 \times (1-\alpha)\%$ confidence interval for $\mu_1 + \mu_2$.

   (b) Consider a two-way classification model

$$y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij},$$

   where $\epsilon_{ij} \sim$ i.i.d. $N(0, \sigma^2)$, $i = 1, 2, 3$ and $j = 1, 2$.

   (i) Derive the best linear unbiased estimator of $\alpha_1 - \alpha_2$.

   (ii) Derive a test for the hypothesis $\alpha_1 = \alpha_2$.

[(7+7) + (4+6) = 24]

Group B

Operations Research

5. (a) Formulate a two-person zero-sum game as a linear programming problem using minimax theorem.

   (b) Find the maximum and minimum distance of the curve $3x^2 + 4xy + 6y^2 = 140$ from the the origin.

(c) Suppose that $S \subseteq R^n$ is a convex set, and $f : S \to R$. Let the epigraph of the function f be defined as $epi\ (f) = \{(x, y) : y \geq f(x), x \in S, y \in R\} \subseteq R^{n+1}$. Show that f is a convex function if $epi\ (f)$ is a convex set. Is the converse true? Justify.

[6+8+(6+4)= 24]

6. (a) Which of the following sets are convex? - Justify.
(i) $X = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid 9x_1^2 + 4x_2^2 \geq 36\}$
(ii) $X = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid x_1 x_2 \geq 0\}$
(iii) $X = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid x_1 + 2x_2 \leq 5\}$
(iv) $X = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid |x_1| + |x_2| \geq 1\}$

(b) Consider the linear programming problem

$$\begin{aligned} \text{Maximize} \quad -\alpha x_1 \ &+ \ x_2 \\ \text{subject to} \quad x_1 + x_2 \ &\leq \ 2 \\ x_1 + 3x_2 \ &\leq \ 3 \\ x_1, x_2 \ &\geq \ 0. \end{aligned}$$

If $(2, 0)$ is the only optimal solution, find the range of α.

(c) Find the number of optimal solution(s) of the linear programming problem with justification

$$\begin{aligned} \text{Maximize} \quad z = 4x_1 \ &+ \ 6x_2 \\ \text{subject to} \quad x_1 + x_2 \ &\leq \ 8 \\ 2x_1 + 3x_2 \ &\geq \ 18 \\ x_1 \ &\geq \ 0 \\ x_2 \ &\text{is} \ \text{unrestricted.} \end{aligned}$$

[6+8+10 = 24]

Reliability

7. Consider a series system with two components C_1 and C_2 with respective lifetimes T_1 and T_2. The joint density function of T_1 and T_2 is given by

$$f(t_1, t_2) = \frac{1}{8} t_1 e^{-(t_1+t_2)/2}, \quad t_1 > 0, t_2 > 0.$$

   (i) Find the reliability function of the system. Hence or otherwise find the mean lifetime of the system.

   (ii) Check whether the system lifetime distribution is IFR.

[(12+4)+8 = 24]

8. (a) Twenty identical units were placed on a life test until ten failures were observed. The following failure times (in hours) were observed

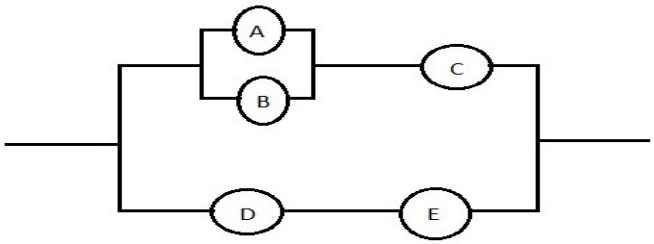
$$55, 20, 35, 95, 40, 100, 5, 15, 70, 65$$

Suppose the lifetimes of the units follow independent and identical exponential distribution.

   (i) Find the maximum likelihood estimate (MLE) of the mean time to failure of the units.

   (ii) Find the MLE of reliability of a unit at 150 hours.

(b) Consider a system with five components A, B, C, D and E. The reliability block diagram of the system is given below.



(i) Find the minimal path sets and the minimal cut sets.

(ii) Write the structure function of the system using either the minimal path sets or the minimal cut sets.

(c) Consider two repairable systems (System 1 and System 2) whose failures occur according to independent Poisson processes with respective rates λ_1 and λ_2. Given that a total of n failures have occurred in the interval $[0, t]$, find the expected number of failures of System 1 in $[0, t]$.

[(6+2)+(4+4) +8 =24]

Statistical Quality Control

9. (a) Describe the data collection procedure for maintaining the average and range control charts to be used for monitoring a quality characteristic of a manufacturing process. Also, describe the procedure for computation of control limits for these two charts.

(b) Discuss type I and type II errors related to control chart. What are the practical implications of these errors in the process?

(c) Consider an injection molding process producing a particular type of plastic part. A control chart is used to control the proportion non-conforming parts. Suppose k samples each of 100 parts are selected from the process. The average number of non-conforming items in k samples together is observed as 10.

(i) Find the three-sigma control limits for the proportion non-conforming chart.

(ii) Suppose that maximum 10% non-conforming parts can be allowed in any sample of 100 parts. Give your opinion with reasons whether the molding process can be considered as capable or not.

[(4+4)+(4+4)+(4+4) =24]

10. (a) Assume that a quality characteristic X follows a Weibull distribution with scale parameter α and shape parameter β. The upper and lower specification limits on X are specified as U and L, respectively.

(i) Find the probability of non-conformances.

(ii) Find the generalized process capability indices C_p and C_{pk}.

(b) Describe the working procedure of single and double sampling plans for attributes. Discuss the relative advantages and disadvantages of a double sampling plan with respect to a single sampling plan.

(c) Describe briefly EWMA control chart.

[(4+4)+(6+6)+4 =24]

Quality Management

11. (a) Describe the quality management system initially known as QS 9000 and currently known as IATF 16949.

(b) Explain Taguchi's definition of 'Quality' with an example.

(c) Which component of Cost of Quality has significant impact on profitability of the organisation? Explain.

[7+5+12=24]

12. (a) Explain the role of Measurement System Analysis (MSA) in Six Sigma DMAIC Methodology.

(b) What are the different phases of Six Sigma DMAIC methodology where control chart can be used? Explain the objectives of using control chart in each of these phases.

[12+12=24]
