JRF (Quality, Reliability & Operations Research): 2016
INDIAN STATISTICAL INSTITUTE

INSTRUCTIONS

The test is divided into two sessions (i) Forenoon session and (ii) Afternoon session. Each session is for two hours. For the forenoon session question paper, the test code is **MMA** and for the afternoon session question paper, the test code is **QRB**. Candidates appearing for JRF (QROR) should verify and ensure that they are answering the right question paper.

For test **MMA**, see a different Booklet. For test **QRB**, refer to this Booklet only.

The test **QRB** is of short answer type. It has two groups. A candidate has to answer **two** of the **four** questions from Group A and **three** of the **eight** questions from Group B.

OUTLINE OF THE SYLLABUS FOR QRB

The syllabus for JFR (QROR) will include the following subject groups:

1 Group A

i) Mathematics
ii) Statistics (including Probability)

2 Group B

i) Operations Research
ii) Quality Management
iii) Reliability
iv) Statistical Quality Control

A broad coverage for each of the above subject groups is given below.

A. **Mathematics**: Algebra, Calculus, Set theory, Vectors and matrices (all at B.Sc. level).

B. **Statistics**: Elementary probability and distribution theory, Bivariate distributions, Multivariate normal distribution, Regression and linear models, Estimation, Test of hypothesis, Design of experiments (block design, full and fractional factorial designs), Markov chain.

C. **Operations Research**: Linear programming (basic theory, simplex algorithm and its variants, duality theory, transportation and assignment problem), Non-linear programming-basic theory, Game theory (two person zero-sum game).

D. **Quality Management** : Quality concepts, Quality costs, Total Quality Management, Six Sigma.

E. **Reliability**: Coherent systems and system reliability, Hazard function, Failure time distribution, Censoring schemes, Estimation and testing in reliability, Replacement models, Repairable system.

F. **Statistical Quality Control**: Statistical process control - attribute and variable control charts, Control chart with memory (CUSUM, EWMA etc.) (univariate only), Multivariate control chart, Process capability analysis, QC tools, Acceptance sampling.

Sample Questions for Group A (Mathematics)

1. (a) Let $f(x) = 3x^4 + 4x^3 - 2x^2 - 4x + 5$ for $x \in [-1, 1]$. Find the maximum value of f in the interval [-1, 1]. Is the function f convex on the interval [-1, 1]?

   (b) Let $a_n > 0$ for $n \geq 1$. Suppose $\sum_{n=1}^{\infty} a_n$ is divergent. Show that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is divergent.

2. (a) Let B be a nonsingular matrix of order n and $D = B'B$. Show that if $x'Dx = 0$, then $x = 0$.

   (b) Let $f : Z \rightarrow R$ where Z is the set of all integers and R is the set of all real numbers such that $f(n)+f(n-1)+f(n-2) = 0$ for all $n \in Z$ and $f(0) = 1$. Find the value of $\sum_{k=1}^{101} f(k)$.

   (c) Consider the set
   $A = \{u = 15x + 21y : x \text{ and } y \text{ are integers}, 0 \leq u \leq 4000\}$.
   Find the number of elements in A.

3. (a) $\{x_n\}$ and $\{y_n\}$ are two sequences of real numbers such that $2x_n^2 + x_n y_n + 2y_n^2 \rightarrow 0$ as $n \rightarrow \infty$. Show that $x_n \rightarrow 0$ and $y_n \rightarrow 0$.

   (b) Let A be a 3×3 real matrix such that $A^2 = 0$. What is the maximum possible rank of A?

   (c) Prove that if $a_1, a_2, ..., a_n$ are positive numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2.$$

4. (a) Consider the sequence $\{(5^n + 4^n)^{\frac{1}{n}},\ n = 1, 2, \ldots\}$. Show that the sequence is bounded and decreasing.

   (b) Let $A \in R^{m \times n}$ and $D = AA^t$. Define

$$\tilde{A} = \begin{bmatrix} D & -A \\ A^t & 0 \end{bmatrix}.$$

   Show that $x^t \tilde{A} x \geq 0\ \forall x \in R^{m+n}$.

(c) Find the global minimizers (if any) for the following functions:

$$i)\ f(x,y) = e^{x-y} + e^{y-x}, \text{ and}$$
$$ii)\ f(x,y) = e^{x-y} + e^{x+y}.$$

5. (a) Let Q denote the set of rational numbers. For given constants a, b, c, define the function f on real numbers:

$$f(x) = \begin{cases} ax & \text{if } x \in Q \\ c - bx & \text{if } x \notin Q. \end{cases}$$

Derive necessary and sufficient conditions on the constants a, b, c for each of the following cases:

i) f is a continuous function,
ii) f is continuous exactly at one point and
iii) f is nowhere continuous.

(b) Let

$$g(t) = \int_0^t \frac{1}{x} \sin\left(\frac{tx}{\pi}\right) dx, \quad \text{for } t > 0.$$

Find $g'(\pi)$.

(c) Find the total number of functions $f : \{1, 2, \ldots, 7\} \to \{-1, 1\}$ satisfying

$$\sum_{i=1}^{7} f(i) > 1.$$

Sample Questions for Group A (Statistics)

1. Suppose die A has 4 red faces and 2 green faces while die B has 2 red faces and 4 green faces. Assume that both the dice are unbiased. An experiment is started with the toss of an unbiased coin. If the toss results in a Head, then die A is rolled repeatedly while if the toss of the coin results in a Tail, then die B is rolled repeatedly. For $k = 1, 2, 3, \ldots$, define X_k to be the indicator random variable such that X_k takes the value 1 if the k-th roll of the die results in a red face, and takes the value 0 otherwise.

   i) Find the probability mass function of X_k.

   ii) Calculate the correlation between X_1 and X_7.

2. Fold over a 2^{5-2} design to construct a 2^{6-2} design. Write the complete defining relation of the resulting design. What is its resolution? Is the resulting design a minimal design?

3. Let $X_{(1)}$, $X_{(2)}$ and $X_{(3)}$ be the order statistics of a random sample of size 3 from

$$f_\theta(x) = \begin{cases} \frac{1}{\theta}, & \text{if } 0 \leq x \leq \theta \\ 0, & \text{otherwise,} \end{cases}$$

   for $0 < \theta < \infty$. Let $T_1 = \alpha_1 X_{(1)}$, $T_2 = \alpha_2 X_{(2)}$ and $T_3 = \alpha_3 X_{(3)}$ be unbiased estimators of θ where α_1, α_2 and α_3 are suitable positive numbers. Find the values of α_1, α_2 and α_3. Compare the efficiencies of T_1, T_2 and T_3.

4. Consider independent random variables $U_1 \sim N(\theta, 1)$, $U_2 \sim N(\theta, 1)$ and $U_3 \sim N(0, 1)$. Suppose these variables are unobservable, but $X = U_1 + U_2$ and $Y = U_2 + U_3$ are observed.

   (a) Find the best linear unbiased estimator (BLUE) of θ based on X and Y.

   (b) Find the variance of the BLUE obtained in part (a).

   (c) If independent paired samples (X_i, Y_i), $i = 1, \ldots, n$ are generated as above, find the BLUE of θ based on the $2n$ observations.

(d) Provide an exact confidence interval for θ with coverage probability 0.95, based on the observations mentioned in part (c).

5. (a) A secretary goes to work following one of the three routes A, B and C. Her choice of route is independent of weather. If it rains, the probability of arriving late following A, B and C are 0.06, 0.15 and 0.12, respectively. The corresponding probabilities if it does not rain are 0.05, 0.10 and 0.15. Assume that, on an average, one in every four days is rainy.

   i) Given that she arrives late on a day without rain, what is the probability that she took route C?

   ii) Given that she arrives late on a day, what is the probability that it was a rainy day?

   (b) Let (X, Y) have the joint probability density function given by

$$f(x,y) = \begin{cases} 24xy & \text{for } 0 < x, y, x + y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the correlation coefficient between X and Y.

6. (a) Consider the linear model

$$Y_i = \beta \frac{i}{n} + \epsilon_i, \; i = 1, \ldots, n,$$

   where β is the unknown regression parameter of interest and ϵ_i's are independent error variables satisfying $E(\epsilon_i) = 0$ and $Var(\epsilon_i) = \sigma^2$ (unknown). Find Best Linear Unbiased Estimator of β.

   (b) Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

   i) Calculate $P[X_3 = 1 | X_0 = 0]$.

ii) Check whether the chain is irreducible.

7. (a) Consider a random sample of size 1 from the population with density

$$f(x;\theta) = 2(\theta - x)/\theta^2, \ 0 \leq x < \theta, \ \theta > 0.$$

   i) Obtain the maximum likelihood estimate of θ (by checking if it maximizes the likelihood).

   ii) Find an unbiased estimator of θ.

   (b) The average yield of grain in an experiment with treatment A is claimed to be greater than that with treatment B. Describe a statistical test for this claim, based on yield data from n_A and n_B experiments with treatment A and B, respectively. Clearly specify the null and alternative hypotheses and the underlying assumptions needed.

Sample Questions for Group B (Operations Research)

1. (a) Consider the following problem:

$$\text{minimize } c'x, \text{ subject to } Ax \geq b, x \geq 0,$$

where $A \in R^{m \times n}, c \in R^n$ and $b \in R^m$.

   i) Show that the optimal solution to this linear programming problem is equivalent to solving a system of equations in non-negative variables.

   ii) If the optimal solution is not unique then show that there cannot be finitely many optimal solutions to the linear programming problem.

   (b) The set of all feasible solutions to a linear programming problem is a convex set.

2. (a) If the i^{th} row of the payoff matrix of an $m \times n$ rectangular game be strictly dominated by a convex combination of the other rows of the matrix, then show that the deletion of the i^{th} row from the matrix does not affect the set of optimal strategies for the row player.

   (b) Let $f : R^n \to R$ be pseudoconvex at $\bar{\mathbf{x}}$. Then show that $\bar{\mathbf{x}}$ is a global minimum if and only if $\nabla f(\bar{\mathbf{x}}) = 0$.

   (c) Show that a balanced transportation problem always has a feasible solution.

3. The working hours of a BPO company which operates round the clock are divided into contiguous blocks of 4 hours each (see the block timings in the table below). Each employee can report for the duty only at the beginning of any block, and work for 8 continuous hours. Based on the work load, the number of employees required in block i is at least r_i, $i = 1, 2, ..., 6$. The values of r_i are given in the table below.

Block (i)	Timings	r_i
1	0300-0700	2
2	0700-1100	10
3	1100-1500	14
4	1500-1900	8
5	1900-2300	10
6	2300-0300	3

The manager of the company wishes to run the operations with minimum number of employees.

(a) Formulate the problem as an optimization problem.

(b) Check if (0, 14, 0, 8, 2, 2) is a feasible solution.

(c) Using duality, solve the problem.

4. (a) Consider the problem:

$$\text{minimize } \frac{p^t x + \alpha}{q^t x + \beta}, \quad q^t x + \beta \neq 0$$

$$\begin{aligned} \text{subject to } Ax &= b \\ x &\geq 0, \end{aligned}$$

where $p, q \in R^n$, $b \in R^m$, $A \in R^{m \times n}$ and $\alpha, \beta \in R$. Formulate the above problem as linear programming problem.

(b) Let $A \in R^{m \times n}$. Show that exactly one of the following two systems has a solution:

$$\begin{aligned} &\text{System 1}: \quad Ax < 0 \text{ for some } x \in R^n \\ &\text{System 2}: \quad A^t y = 0, \ y \geq 0 \text{ for some } y \in R^m. \end{aligned}$$

(c) Solve the following problem:

$$\begin{aligned} \text{maximize } f(x_1, x_2, x_3) &= x_1 x_2^2 x_3 \\ \text{subject to } x_1 + x_2 + x_3^2 &= k, \end{aligned}$$

where $k > 0$ is fixed and x_1, x_2, x_3 are positive real numbers.

Sample Questions for Group B (Quality Management)

1. (a) A process has five operation steps $S_i, i = 1(1)5$. Defects obtained in each step and the number of units produced over a fixed time period are denoted by D_i and U_i respectively. Explain the procedure for finding out defects per unit (DPU_i), throughput yield at each step, rolled throughput yield and total defects per unit using rolled throughput yield with an underlying Poisson distribution.

   (b) Outline the procedure for obtaining normalized yield and DPU_{Norm} for the process described in (a) above.

   (c) Considering a 1.5σ shift, explain the procedure for obtaining the $Z_{Short-Term}$ and Z_{Norm} for the same process.

2. (a) Consider the manufacture of a mechanical component used in a copier machine. The parts are manufactured in a machining process at a rate of approximately 100 parts per day. For various reasons the process is operating at a first-pass yield of about 75%. About 60% of the fallout can be reworked into an acceptable product, and the rest must be scrapped. The direct manufacturing cost is approximately Rs. 20 per part. Parts that can be re-worked incur an additional processing charge of Rs. 4. Find the manufacturing cost per good part produced.

   (b) An engineering study of the process described in (a) above reveals that excessive process variability is responsible for the extremely high fallout. A new statistical process control procedure is implemented that reduces the variability and consequently the process fallout decreases from 25% to 5%. Of the 5% fallout produced, about 60% can be reworked, and the remaining 40% are scrapped. What will be the manufacturing cost per good part produced after the process control program is implemented? How much will be the % reduction in manufacturing cost?

Sample Questions for Group B (Reliability)

1. (a) The UN security council consists of five permanent members having 7 points each and ten temporary members having 1 point each. In order to get a resolution passed, at least 39 points are needed in favour. Present this problem (of getting a resolution passed) as a coherent system with 15 components and obtain the structural importance of the components. Find a non-trivial modular decomposition of the system.

   (b) Prove that the min path sets of a coherent system are the min cut sets of its dual and vice versa.

2. (a) The hazard rate for an item is given by

$$h(x) = \begin{cases} a, & \text{if } 0 < x \leq x_0 \\ a + b(x - x_0), & \text{if } x > x_0, \end{cases}$$

   where a, b, x_0 are positive constants. Derive the reliability function. For known x_0 and based on random right censored lifetime data, write down the likelihood function for estimating a and b.

   (b) Consider the Weibull distribution with reliability function

$$R(t) = e^{-\alpha t^\beta}, t > 0, \alpha > 0, \beta > 0.$$

   i) When does the distribution have decreasing failure rate?

   ii) Obtain an expression, possibly in the form of an integral, for the mean remaining life function for this distribution.

3. (a) Suppose the n components of a parallel system have independent exponential lifetimes with failure rates as $\lambda_1, \cdots, \lambda_n$. Compute the system reliability function and the mean system lifetime.

   (b) Suppose that two independent systems, with identical and exponentially distributed lifetime with failure rate 3×10^{-7} per hour, are either to be placed in active parallel or a standby configuration. For $t = 7$ years, derive the reliability gain of one over the other configuration.

4. (a) Consider a series system with two independent components each with strength having exponential distribution with parameter α. The system is subjected to a random stress following exponential distribution with parameter β. A component fails when the stress exceeds its strength. Find the reliability of the system.

   (b) Suppose n independent units each having *exponential* (λ) life distribution are put on a life test for a pre-fixed time T_0. Find the distribution of the number of failures during the test. Derive the maximum likelihood estimate of expected number of such failures.

5. Consider a coherent system with four independent components 1, 2, 3 and 4, in which the min path sets are $\{1,3\}$, $\{2,3\}$ and $\{4\}$.

   (a) Write down the structure function of the system.

   (b) Write down the min cut sets of the system.

   (c) If the lifetime of the ith component follows the uniform distribution over $[0, i]$ for $i=$ 1, 2, 3, 4, find the reliability of the system at time $t > 0$.

   (d) Prove that the reliability of the above system at time t is less than or equal to

$$1 - \prod_{i=1}^{4} \left[\frac{\min\{t, i\}}{i} I(t > 0) \right],$$

   where $I(\cdot)$ denotes the indicator function.

6. (a) An equipment is replaced by a spare in the event of its failure or having reached the age of T_0, whichever is earlier. Let F be the lifetime distribution of the equipment. Derive the expectation of this replacement time.

   (b) Consider a one-unit device and suppose that there are n spares stocked at time 0. When the operating unit fails, it is replaced by a spare having independent and identical life distribution given by the density $f(x) = \lambda e^{-\lambda x}$, $\lambda > 0, x > 0$. Assume that all the replacements are instantaneous. Derive an expression to determine the minimum value of n such that the probability of uninterrupted service over time interval $(0, t]$ is more than 0.95.

Sample Questions for Group B (Statistical Quality Control)

1. (a) For a single sampling plan with curtailed inspection associated with rejection, derive the expression for ASN as a function of product quality.

   (b) Derive the following properties of the binomial OC function of a single sampling plan with sample size n and acceptance number c at a process average p:

$$B(c+1, n+1, p) - B(c, n, p) = qb(c+1, n, p)$$

   where

$$B(c, n, p) = \sum_{x=0}^{c} b(x, n, p) = \sum_{x=0}^{c} \binom{n}{x} p^x (1-p)^{n-x}.$$

2. (a) Show that if $\lambda = 2(w+1)$ for the EWMA control chart, the chart is equivalent to a w period moving average control chart in the sense that the control limits are identical for large t.

   (b) For a control chart of a normally distributed quality characteristic if all of the next seven points fall on the same side of the center line, we conclude that the process is out of control. (i)What is the α risk? (ii)If the mean shifts by one standard deviation and remains there during collection of the next seven samples then find what is the β risk associated with this decision rule.

3. (a) Describe any one method for estimating the process capability indices C_p and C_{pk} when the quality characteristic is a non-normal variable.

   (b) A process operates at the mid-specification for a variable quality characteristic. Show that the extent of off-specification and the associated loss will be minimum assuming that the natural variation of the process is more than the desired engineering tolerance and the losses incurred per unit on both sides of the specification limits are identical.

4. (a) Describe a procedure for estimating process capability when each subgroup consists of an individual unit.

(b) In designing a fraction nonconforming chart with centre line at $\bar{p} = 0.20$ and 3-sigma limits, (i) what sample size (n) is required to yield a positive lower control limit and (ii) what value of n is necessary to obtain a probability of 0.50 for detecting a shift in the process $\bar{p}$ to 0.26?