1. Define a government to be *Utilitarian* if it chooses its actions in order to maximise the sum of utilities of individuals under it, and a government to be *Rawlsian* if it chooses its actions in order to maximise the utility of that individual under it who is in the most disadvantaged position.

   Suppose a society consists of 3 individuals and a government. Preferences of individual i are given by the utility function $u_i(I_i, l_i)$, where I_i is post-tax income and l_i is the amount of labour supplied, $i \in \{1, 2, 3\}$. Individuals earn wage income in exchange for labour supplied with individual i facing wage rate w_i. Assume $0 < w_1 < w_2 < w_3$, so that individual 1 is in the most disadvantaged position.

   The government wishes to impose a purely redistributive tax schedule (where the government's total tax revenue is 0), such that individual i pays tax $T_i = cw_il_i + d$, so that her post-tax income satisfies $I_i = w_il_i - T_i$. The government chooses the tax schedule, i.e., the values of c and d, optimally by maximising its own utility function.

   Consider monotone transformations of utility functions of all individuals, (f_1, f_2, f_3), and let $u_i^{'} \equiv f_i(u_i)$. Specifically consider the following three classes:

$$\text{(A) } f_i(u_i) = a_i + bu_i$$
$$\text{(B) } f_i(u_i) = a + b_iu_i$$
$$\text{(C) } f_i(u_i) = a + bu_i$$

   (a) For each class, determine if a Utilitarian government's optimal tax schedule is unchanged under the transformed profile of utility functions, $(u_1^{'}, u_2^{'}, u_3^{'})$. [2+2+2]

   (b) For each class, determine if a Rawlsian government's optimal tax schedule is unchanged under the transformed profile of utility functions, $(u_1^{'}, u_2^{'}, u_3^{'})$. [2+2+2]

(c) Now suppose the the utility functions take the form

$$u_i = w_i l_i - \frac{l_i^2}{2},\ i \in \{1,2,3\}$$

Find the values of c and d under Rawlsian and Utilitarian governments. [7+6]

2. There are two *types* of workers, $\theta \in \{1,2\}$. A worker knows his own type, and is type 1 with probability $\pi \in (0,1)$ and type 2 with probability $1-\pi$. A worker of type θ can purchase $e \geq 0$ units of education at cost $c(e) = 2\frac{e^2}{\theta}$, and has marginal product θ if he works for a firm. There are multiple firms, which do not observe the type of any worker. These compete perfectly for a given worker, so that the equilibrium wage equals the expected marginal product of the worker given the firms' beliefs.

(a) Find all strategies or choices of education as a function of type for workers that can be part of a separating equilibrium in pure strategies of this game. For one such strategy, state a belief function for the firms that forms a separating sequential equilibrium with the chosen strategy. [6+2]

(b) Find all levels of education that the worker can choose as part of a pooling equilibrium in pure strategies of this game. For one such strategy, state a belief function for the firms that forms a pooling sequential equilibrium with the chosen strategy. [5+2]

(c) Suppose that the worker anticipates that his wage will never lie outside the range $[1,2]$. Identify all strictly dominated choices of education for the low ($\theta = 1$) type of worker, as well as for the high ($\theta = 2$) type of worker. [3+2]

(d) Suppose now that if the worker chooses a level of education that is strictly dominated if he is the low type, but not strictly dominated if he is the high type, the firms must believe that the worker is the high type with probability 1. Which education levels that you found in part (b) above can be chosen in a pooling sequential equilibrium under these restrictions on beliefs? [5]

3. Consider a closed economy with three goods - output, money and labour - and three sectors - the aggregate household, the aggregate firm and the government. Let w and p respectively denote the wage rate of labour (supplied by the household) and the price of output (produced by the firm).

   Further, let y denote the output of the firm, and $F(l)$ denote its production function, where $F^{'}(l) > 0, F^{''}(l) < 0$, and l is labour employed. Suppose the firm perceives that demand for its output is $Ap^{-\alpha}$, $A > 0$ and $\alpha > 1$, and chooses y to maximize its profit $\pi = py - wl$ subject to $y = F(l)$ and $y = Ap^{-\alpha}$ (technology and demand constraints respectively).

   $U = \gamma \log(c) + (1-\gamma)\log(\frac{M}{p}) - V(l)$ is the household's utility function, where c is the consumption of the output, M is the holding of money, and $V(l)$ is the disutility from the supply of labour, with $\gamma \in (0,1), V^{'}(l) > 0, V^{''}(l) > 0$. The household perceives that demand for its labour is $Bw^{-\beta}$, $B > 0$ and $\beta > 1$, and its budget constraint is given by $pc + M = wl + \pi + M_0 - pt$, where t denotes the real taxes paid by the household to the government, M_0 is its initial money holding, and π is the profit of the firm which is rebated back lump sum to the household. Suppose the household chooses c, M, and l to maximize utility, subject to the budget constraint $pc + M = wl + \pi + M_0 - pt$ and the labour demand constraint $l = Bw^{-\beta}$.

   Denoting the government's demand for output by g, the government's budget constraint is given by $M - M_0 = p(g - t)$.

Suppose that output market equilibrium is given by $y = c + g$, and assume that prices and wages are flexible.

(a) Show that the firm's maximization problem implies that

$$\left(\frac{\alpha-1}{\alpha}\right)\frac{p}{w} = \frac{1}{F'(l)},$$

and interpret this equation. [4+1]

(b) Now show that employment (l) and output (y) are independent of money holding (M_0) in equilibrium. [10]

(c) Further show that given g and t, the equilibrium price p changes proportionately with M_0, assuming a positive p exists. [10]

4. (a) A process Y_t is defined as follows:

$$Y_t = \alpha + \beta t + \varepsilon_t - \varepsilon_{t-1},$$

where ε_t follows white noise process with mean 0 and variance σ^2. Find the auto-correlation function of ∇Y_t, where $\nabla Y_t = Y_t - Y_{t-1}$. [5]

(b) Let $Y_t = \phi Y_{t-1} + \varepsilon_t$, where $|\phi| < 1$, and ε_t follows white noise process with mean 0 and variance σ^2. Show that

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty.$$

where $\gamma(h)$ is the auto-covariance function between Y_t and Y_{t-h}, and h is an integer. [5]

(c) Let $\{X_1, X_2, \cdots, X_n\}$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. Consider the following point estimators of σ^2

- $\hat{\sigma}_1^2 = \dfrac{1}{n-3}\sum_{i=1}^{n}(X_i - \bar{X})^2$
- $\hat{\sigma}_2^2 = (X_1 - \bar{X})^2$

where $\bar{X}$ is the sample mean. Answer the following questions with justification:

i. Which are the unbiased estimators? [5]

ii. Which are the consistent estimators? [5]

iii. Which is the most efficient? [5]