1. For any vector $\mathbf{x} \equiv (x_1, \ldots, x_n) \in \mathbb{R}^n$, denote the *increasing* and *decreasing* rearrangements of $\mathbf{x}$ as

   - decreasing: $(x_{[1]}, \ldots, x_{[n]})$ where $x_{[1]} \geq \ldots \geq x_{[n]}$
   - increasing: $(x_{(1)}, \ldots, x_{(n)})$ where $x_{(1)} \leq \ldots \leq x_{(n)}$

   For any pair of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we define a relation between them. We say $\mathbf{y}$ *majorizes* $\mathbf{x}$ if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \qquad \forall\, k \in \{1, \ldots, n-1\}$$
$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$$

   In this case we write $\mathbf{y} \succ \mathbf{x}$, and refer to $\succ$ as the majorization relation. For instance, if $n = 3$ and $\mathbf{x} = (0.2, 1, 0)$ and $\mathbf{y} = (1.2, 0.1, -0.1)$, then $y_{[1]} = 1.2 > 1 = x_{[1]}$, $y_{[1]} + y_{[2]} = 1.2 + 0.1 = 1.3 > 1 + 0.2 = x_{[1]} + x_{[2]}$, and $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$ implies $y \succ x$.

   (a) Give an example of a pair of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ such that $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$, but neither $\mathbf{x}$ majorizes $\mathbf{y}$ nor $\mathbf{y}$ majorizes $\mathbf{x}$. [2]

   (b) Consider any three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ such that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = \sum_{i=1}^n z_i$. Show that the majorization relationship is *transitive*, i.e., if $\mathbf{x} \succ \mathbf{y}$ and $\mathbf{y} \succ \mathbf{z}$, then $\mathbf{x} \succ \mathbf{z}$. [2]

   (c) Consider the following vectors in $\mathbb{R}^n$:
   $\bar{\mathbf{a}} = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$, $\mathbf{a}_1 = (1, 0, \ldots, 0)$ and
   $\mathbf{a} = (a_1, \ldots, a_n), a_i \in [0, 1] \forall i \in \{1, \ldots, n\}$ and $\sum_{i=1}^n a_i = 1$
   Compare $\bar{\mathbf{a}}$, $\mathbf{a}$, and $\mathbf{a}_1$ using the majorization relation (i.e., find if any of these vectors majorizes any of the other vectors). [10]

(d) Take any vector $\mathbf{x} \equiv (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ with $\sum_{i=1}^n x_i \neq 0$. Consider the *equalization* of $\mathbf{x}$ as the vector

$$\bar{\mathbf{x}} := \Big(\frac{x_1 + \ldots + x_n}{n}, \ldots, \frac{x_1 + \ldots + x_n}{n}\Big)$$

Compare $\mathbf{x}$ and $\bar{\mathbf{x}}$ using the majorization relation. [5]

(e) Show that $\mathbf{y} \succ \mathbf{x} \iff$

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \qquad \forall\, k \in \{1, \ldots, n-1\}$$
$$\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$$

[6]

2. (a) Suppose a, b, and c are in Geometric Progression. If the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, show that $\frac{d}{a}$, $\frac{e}{b}$, and $\frac{f}{c}$ are in Arithmetic Progression. [8]

(b) Find a and b if $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$. [4]

(c) Consider a function $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = 0$ if $x = 0$ and, for $x \neq 0$,

$$f(x) = \begin{cases} xcos(\frac{1}{x}), & \text{for } x \in \mathbb{Q}; \\ 0, & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that the f is continuous at $x = 0$. [7]

(d) Consider a pair of continuous functions $f, g : [0, 1] \to \mathbb{R}$ such that $\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0, 1]\}$. Show that there exists $c \in [0, 1]$ such that $f(c) = g(c)$. [6]

3. (a) A *simple random sampling without replacement* (**SRSWOR**) technique refers to drawing a sample of n units from a population of N units in such a way that

each unit of the population has an equal and independent chance of being included in the sample without replacing the units in the population which are already selected in the sample.

Suppose $(y_1, \ldots, y_n)$ is a sample generated by SRSWOR. Denoting the population as $(Y_1, \ldots, Y_N)$, we have:

- Sample mean $\bar{y}_n = \frac{1}{n}\sum_{i=1}^{n} y_i$
- Population mean $\bar{Y}_N = \frac{1}{N}\sum_{i=1}^{N} Y_i$
- Sample mean square $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y}_n)^2$
- Population mean square $S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y}_N)^2$
- Population variance $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y}_N)^2$

Some properties of SRSWOR are as follows:

- The sample mean is an unbiased estimate of the population mean: $\mathrm{E}(\bar{y}_n) = \bar{Y}_N$
- The sample mean square is an unbiased estimate of the population mean square: $\mathrm{E}(s^2) = S^2$
- The variance of the sample mean is given by:
$$V(\bar{y}_n) = \mathrm{E}(\bar{y}_n^2) - \bar{Y}_N^2 = \left(\frac{1}{n} - \frac{1}{N}\right)S^2$$
- If a sample of size n is drawn from a population of size N and variance σ^2, and if y_i and y_j, $i \neq j$, are any two members of the sample, then $V(y_i) = V(y_j) = \sigma^2$.

For a survey to estimate health expenditure of a state, a household sample of size n is drawn from a homogenous population having N units by SRSWOR. A sub-sample of n_1 units is then drawn from the first sample of n units, again by SRSWOR. Let $\bar{y}_1$ be the sample mean based on the second sample of size n_1, and $\bar{y}_2$ be the sample mean based on the residual sample of size $n_2 = n - n_1$. Consider the estimator of the population mean $(\bar{Y}_N)$ given by

$$\hat{\bar{Y}}_N = w\bar{y}_1 + (1-w)\bar{y}_2$$

i. Show that $\mathrm{E}(\widehat{\bar{Y}}_N) = \bar{Y}_N$. [5]

ii. What is $V(\widehat{\bar{Y}}_N)$? [7]

iii. Find the optimal value of w, i.e., the value for which $V(\widehat{\bar{Y}}_N)$ is minimum. [5]

iv. Find the values of $\widehat{\bar{Y}}_N$ and $V(\widehat{\bar{Y}}_N)$ at the optimal value of w you have identified above. [1+2]

(b) If $P(t)$, the population at time t is governed by the differential equation $\frac{dP(t)}{dt} = rP(t)$ (i.e., the population change depends only on a fixed growth rate and the size of the population), then derive an expression for $P(t)$ in terms of population in some base year $P(0)$, r and t. Find an expression for the time required for the population to double in size. [3+2]

4. (a) There are three coins in a basket. The probability of getting head for these coins are 0.1, 0.5 and 0.7, respectively. A coin is randomly selected from the basket and then flipped ten times. Let N be the number of heads obtained in these flips.

i. Find $P(N = 0)$. [5]

ii. Find the distribution of N. [5]

(b) Let $\mathrm{E}(Y|X) = \alpha + \beta X$, where α and β are constants. Show that

$$\beta = \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(X)}$$

[5]

(c) The number of customers entering a store on a given day is Poisson distributed with mean $\lambda = 10$. The amount of money spent by a customer is uniformly distributed over $[0, 100]$. Find the mean and variance of the amount of money that the store takes in on a given day. [5+5]