1. (a) Let $f : [0,1] \to [0,1]$ be a function such that $f(x) = 1 - \sqrt{1-x^2}$. Show that $1 - f(1 - f(x)) = x$.

   (b) Let $f : (0,1) \to \mathbb{R}$ be monotonically increasing. Show that $\lim_{x \mapsto c^+} f(x) = \inf\{f(x) \,|\, c < x\}$.

   (c) A firm produces an output y using two inputs x_1 and x_2 as $y = \sqrt{x_1 x_2}$. Union agreements obligate the firm to use at least one unit of x_1 in its production process. The input prices of x_1 and x_2 are given by w_1 and w_2, respectively. Assume that the firm wishes to minimize the cost of producing $\bar{y}$ units of output.
   (i) Set up the firm's cost-minimization problem. Is the feasible set closed? Is it compact?
   (ii) Derive the input demand functions from the cost minimization problem.

   $[5 + 10 + ((3 + 1 + 1) + 5) = 25]$

2. (a) From a group of six men and five women, five persons are to be selected to form a committee so that at least two women are there on the committee. In how many ways can this be done?

   (b) Let α and β be the roots of the equation $x^2 - 4x + A = 0$ and γ and δ be the roots of the equation $x^2 - 36x + B = 0$. If α, β , γ and δ are in geometric progression having positive common ratio, find the values of A and B.

   (c) Show that the function $f(x, y) = \sqrt{xy}$, $x > 0, y > 0$, is concave.

   $[7 + 8 + 10 = 25]$

3. (a) Let A be a real matrix of order $m \times n$ and let $\mathbf{x}$ be a real column vector of length n. If $A\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$, show that $A = \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix.

(b) Let Σ be a variance-covariance matrix of a random vector $\mathbf{X}$ of length p. Show that the determinant of Σ is always greater than or equal to zero.

(c) Show that for any 4×4 real symmetric matrix A, the trace of $(\mathbf{I} + A + A^2)$ is always greater than or equal to 3, where where $\mathbf{I}$ denotes the identity matrix.

(d) Suppose A is an $n \times n$ non-null matrix with $A\mathbf{1}_n = \mathbf{0}$ where $\mathbf{1}_n = (1, 1, \cdots, 1)^T$ is a column vector of length n. Show that the determinant of A is zero.

$[4 + 7 + 7 + 7 = 25]$

4. (a) Suppose in a family, there are four kids. It is given that at least one of them is a girl. What is the probability that the family has at least two girls?

(b) Let X be a random variable taking the value 0 with probability p and the value Y with probability $1 - p$, where Y is a variable following a Poisson distribution with mean λ. Find the expectation and variance of X.

(c) Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2. Define

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2,$$

where $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Show that $\mathbb{E}(S^2) = \sigma^2$, where $\mathbb{E}$ is the expectation operator.

$[5 + 10 + 10 = 25]$