1. Find the value(s) of $\lambda$ for which the following system of linear equations

$$\begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

   (a) has a unique solution,

   (b) has infinitely many solutions,

   (c) has no solution.

[2+4+4]

2. If

$$\int_0^x f(t)dt = x^2 \sin x + x^3,$$

   then find the value of $f(\frac{\pi}{2})$.

[10]

3. For each of the statements below, prove the result if it is true, give a counter example if it is not true.

   Suppose $\sum a_n$ with $a_n > 0$ is convergent. Then,

   (a) $\sum \sqrt{a_n a_{n+1}}$ is convergent,

   (b) for all $0 < \delta < 1$, $\sum a_n^{1+\delta}$ is convergent,

   (c) for all $0 < \delta < 1$, $\sum a_n^{1-\delta}$ is convergent.

[4+3+3]

4. (a) Let $X$ be a Binomial$(5, p)$ random variable and let $P(X = 2) = 2P(X = 3)$. Find the variance of $X$.

   (b) Let $X$ be Poisson$(2)$ and $Y$ be Binomial$(10, \frac{3}{4})$ random variables. If $X$ and $Y$ are independent, then find the value of $P(XY = 0)$.

[3+7]

5. Let $\Re$ be the set of real numbers. Consider $f : \Re \to \Re$ as follows:

$$f(x) = \begin{cases} \min\{|x|, 1 - |x|\} \text{ if } x \text{ is rational and } x \in [-1, 1] \\ 0 \text{ otherwise.} \end{cases}$$

Then, prove or disprove each of the statements below.

   (a) $f$ is continuous at $x = -1$ and $x = 1$,

   (b) $f$ is differentiable at $x = 0$.

[5+5]

6. (a) In how many ways can 900 identical chocolates be divided among 200 children such that each child gets at least 4 chocolates.

   (b) In how many ways can 7 non-identical chocolates be divided among 3 children such that each child gets at least 1 chocolate.

[6+4]

7. Let $\Re, \Re_+$, and $\Re_{++}$ be the sets of real numbers, non negative real numbers, and strictly positive real numbers, respectively. Consider a

two variable function $f : \Re^2_+ \to \Re$ which is differentiable in $\Re^2_{++}$, homogeneous of degree two and, for all $x, y, a, b \in \Re_+$, satisfies $f(x+a, y) = f(x, y) + ay$, $f(x, y+b) = f(x, y) + bx$ and $f(x, 0) = f(0, y) = f(0, 0) = 0$. Using these conditions, identify the exact form of the function $f$.

[10]

8. Show that if $f$ is a continuous real valued function on $[0, 1]$, then there exists a point $c \in (0, 1)$ such that $\int_0^1 xf(x)dx = \int_c^1 f(x)dx$.

[10]

9. In how many ways you can select $6$ letters out of $100A$'s, $80B$'s, $60C$'s, $3D$'s, $2E$'s, and $1F$?

[10]

10. Consider the following polynomial equation

$$x^2 + \alpha x + \ln 2 = 0.$$

Mr. A and Mr. B are drawing $\alpha$ using distributions $N(0, 1)$ and $U(-2\sqrt{2\pi}, 2\sqrt{2\pi})$, respectively, where $N(0, 1)$ denotes the standard normal distribution, and for $a < b$, $U(a, b)$ denotes the uniform distribution over the interval $[a, b]$. We say someone is successful if he chooses $\alpha$ such that the polynomial equation has a real root. Who has a higher probability of being successful?

[10]