Part-I

Mathematical and Logical Reasoning

Answer all questions. Each question carries 5 marks.

1. Let the characteristic polynomial of A be given by $p(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c$, $c > 0$, and the eigenvalues of A be in arithmetic progression.

   (a) Show that A^{-1} exists if and only if $a \neq 0$.

   (b) Find the eigenvalues of A^5.

   [2+3=5]

2. Let $f(x)$ be a real polynomial such that $xf(x-1) = (x-4)f(x)$. Find all such polynomial $f(x)$.

   [5]

3. Let A be a 2×2 matrix with positive entries. Show that A is diagonalizable (i.e., A is equivalent to a diagonal matrix under similarity transformation).

   [5]

4. Evaluate

   $$\iint_S \left(\nabla \times \vec{F}\right) \cdot \hat{n}\, \mathrm{dS},$$

   where $\vec{F} = x\hat{i} + y\hat{j} + (z^2 - 1)\hat{k}$ and S is the surface of the closed cylinder $x^2 + y^2 = 4, z = 0, z = 1$.

   [5]

5. Find the area enclosed by the curves $y = \cos(x), y = x + 1$ and x-axis.

   [5]

6. Find the real and imaginary parts of $\sqrt{i}^{\sqrt{i}}$.

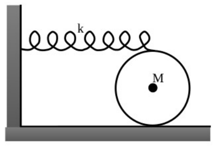
   [2.5+2.5=5]

Part-II

Physics

Answer any five questions. Each question carries 14 marks.

1. (a) The top of a wheel of mass M and radius R is connected to a spring at its equilibrium length with spring constant k, as shown in the figure. If the wheel rolls without slipping,



determine the frequency of small oscillations. Assume the wheel's mass is entirely concentrated at its center.

(b) Synchronized clocks A and B are at rest in an inertial reference frame and are separated by a distance L. A third clock, C, moves with velocity $3c/5$ along the line joining A and B, where c is the speed of light in vacuum. When C passes A, clocks A and C read $t = 0$. What are the readings of clocks B and C when C reaches B?

[7+7=14]

2. A particle of mass m moving in a plane under a central force $F(r) = -\frac{k}{r^2} + \frac{k'}{r^3}$ (assume $k > 0$).

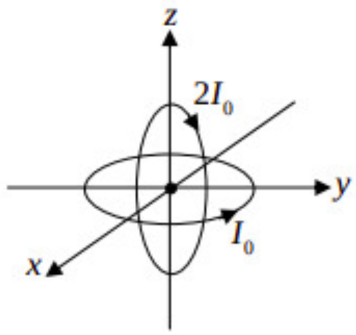
(a) Determine the Lagrangian for this system in terms of the polar coordinates r, θ and their velocities.

(b) Derive the equations of motion for r and θ, and show that the orbital angular momentum l is a conserved.

(c) Assume that $l^2 > -mk'$. Find the equation for the orbit, i.e., express r as a function of θ.

[3+(4+3)+4=14]

3. (a) Two circular current carrying loops, both with radius R, are centered at the origin and placed perpendicular to each other, as shown in the figure. The loop in the xy-plane



carries a current I_0, while the loop in the xz-plane carries a current $2I_0$. Calculate the resulting magnetic field $\vec{B}$ at the origin.

(b) Three point charges, each with charge q, are placed at the vertices of an equilateral triangle. Additionally, a point charge $-Q$ is placed at the centroid of the same triangle. Determine the value of Q in terms of q such that each charge q experiences zero net force.

[7+7=14]

4. (a) A particle is in the first excited state of a one-dimensional box of length L. Suddenly, the box expands to twice its size, leaving the wave function undisturbed. Calculate the probability of the particle being in the ground state if its energy is immediately measured.

(b) Employing first-order perturbation theory, calculate the energy of the first two states for an infinite asymmetric potential well of width 'a', whose portion AB has been sliced off.

[7+7=14]

5. A quantum free particle of mass m is confined in a cubic box of length L with periodic boundary conditions at temperature T.

   (a) Find the partition function Z of the system in the canonical ensemble.

   (b) The density operator of this system is given by $\hat{\rho} = \frac{\exp(-\hat{H}/k_B T)}{Z}$, where $\hat{H}$ is the Hamiltonian of the system, and k_B is the Boltzmann constant. Find the matrix elements of the density operator in three-dimensional co-ordinate basis.

[6+8=14]

6. Consider a one-dimensional periodic lattice of lattice constant a. The solution to the Schrödinger equation for an electron in this lattice is given by the Bloch wavefunction: $\psi_{n\vec{k}}(\vec{r}) = \exp(ikr)u_{n\vec{k}}(\vec{r})$, where $u_{n\vec{k}}(\vec{r})$ is a function with the same periodicity as the lattice, n denotes the band index, and k is the wave vector (assumed to be within the first Brilloin zone).

   (a) Find the Schrödinger equation satisfied by $u_{n\vec{k}}(\vec{r})$ (not $\psi_{n\vec{k}}(\vec{r})$).

   (b) In the presence of a weak periodic potential $V(x) = v\cos(Gx)$, where G is a reciprocal lattice unit vector, determine the energy gap that opens up between the $n = 0$ and $n = 1$ bands at the boundary of the first Brillouin zone using degenerate perturbation theory.

[6+8=14]

7. A quantum particle of mass m moves between two impenetrable concentric spheres of radii $r = a$ and $r = b$ $(b > a)$, both centered at the origin. No additional potential acts on the particle.

   (a) What are the quantum numbers describing the energy eigenstates of the system?

   (b) By separating the wave function into radial and angular parts, write down the Schrödinger equation obeyed by the radial part.

   (c) Determine the ground state energy of the system.

   (d) Find the normalized wave function of the ground state.

[2+2+5+5=14]

8. (a) i. Draw the two Feynman diagrams corresponding to the Bhaba scattering process,

$$e^- + e^+ \to e^- + e^+$$

   within the framework of quantum electrodynamics.

   ii. Find the invariant scattering amplitudes $\mathcal{M}_1, \mathcal{M}_2$ corresponding to the two diagrams.

   (b) i. Prove that the action for the massless Dirac field is invariant under the dilatations:

$$x \to x' = e^{-\rho} x, \quad \psi(x) \to \psi'(x') = e^{3\rho/2}\psi(x)$$

   where ρ is a constant.

   ii. Calculate the Noether current and charge.

[((2+2)+(2+2))+(2+(2+2))=14]

Part-III

Mathematics

Answer any five questions. Each question carries 14 marks.

1. (a) Let $\mathcal{P}_n$ denotes the real vector space (i.e. over $\mathbb{R}$) of polynomials with real coefficients and degree $\leq n$. Write down a matrix representation (with respect to your preferred basis) for the differential operator D defined as

$$D\colon \mathcal{P}_5 \to \mathcal{P}_4, \quad Dp(x) := \frac{\mathrm{d}p(x)}{\mathrm{d}x} = p'(x).$$

   (b) Let the characteristic polynomial of an $n \times n$ real matrix A be given by

$$|\lambda\mathbb{I} - A| = \sum_{i=0}^{n} c_i \lambda^i, \; c_n = 1.$$

   Express $(\lambda\mathbb{I} - A)^{\mathrm{adj}}$ in terms of A, λ, and the c_i's.
   (Here A^{adj} denotes the adjugate of A, *i.e.*, the transpose of the cofactor matrix of A).

[7+7=14]

2. (a) Let X be the set of all positive integers. Show that

$$d(m, n) = \left|m^{-1} - n^{-1}\right|$$

   is a metric on X. Is the metric space (X, d) complete? Justify your answer.

   (b) Prove that a convex function f defined on an open interval $I \subseteq \mathbb{R}$ is continuous. Must a convex function on an arbitrary interval be continuous?

[7+7=14]

3. (a) Let A be a real symmetric matrix and m be a positive integer. Show that the following matrix is non-singular

$$A^{2m} + A^{2m-1} + \mathbb{I}.$$

(b) Let R be the ring of all real-valued continuous functions on $[0, 1]$. Consider the mapping $\phi : R \to \mathbb{R}$ defined by

$$\phi(f) = f\left(\frac{1}{2}\right), \quad \text{for } f \in R.$$

i. Show that ϕ is an onto homomorphism.

ii. Determine the kernel of ϕ.

iii. Deduce that the set $S = \{f \in R \mid f(1/2) = 0\}$ is a maximal ideal in R.

[5+(3+3+3)=14]

4. (a) Find the non-trivial solution of the Sturm-Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < \pi,$$

subject to the boundary conditions $y(0) = 0$ and $\frac{dy}{dx} = 0$ at $x = \pi$.

(b) Linearize the non-linear system described by

$$\frac{dx}{dt} = x + 2y + xy^2, \qquad \frac{dy}{dt} = 2x + y - x^2y$$

around its equilibrium point at the origin and examine the stability of the system.

[8+6=14]

5. (a) Show that for any function $f \in \mathcal{C}[0,1]$ and any number $\epsilon > 0$, there exists a polynomial p, all of whose coefficients are rational numbers, such that $|f - p| < \epsilon$.

(b) i. Find the poles of the complex valued function

$$f(z) = \frac{e^{imz}}{z(z^2 + a^2)}$$

and compute the residue at each pole.

ii. Evaluate $\displaystyle\int_0^\infty \frac{\sin mx}{x(x^2 + a^2)}\, dx.$

[6+(4+4)=14]

6. (a) Let $n \geq 3, H = \{\beta \in S_n \mid \beta(1) = 1 \text{ or } 2 \text{ and } \beta(2) = 1 \text{ or } 2\}$ where S_n is a symmetric group of order n. Show that H is a subgroup of S_n. Hence determine the cardinality of H.

(b) Let $u(x,t)$ be the solution of the Cauchy problem defined by the wave equation,

$$\frac{\partial^2 u}{\partial t^2} - 9\frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad t \geq 0.$$

The initial conditions are given by:

$$u(x,0) = f(x) = \begin{cases} 1, & \text{if } |x| \leq 2, \\ 0, & \text{if } |x| > 2. \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) = \begin{cases} 1, & \text{if } |x| \leq 2, \\ 0, & \text{if } |x| > 2. \end{cases}$$

i. Find $u(0, 1/6)$.

ii. Discuss the large-time behavior of the solution.

iii. Find the maximum value of $u(x,t)$ and the points where the maximum is achieved.

[6+(3+2+3)=14]

7. (a) The velocity field of a fluid flow is given by

$$\vec{v} = (5x)\hat{\imath} + (15y + 11)\hat{\jmath} + (19t^2)\hat{k}\,\mathrm{m/s}.$$

Find the path of a particle which is at $(4, 6, 2)\,\mathrm{m}$ at time $t = 3\,\mathrm{s}$.

(b) Does the following transformation

$$P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p \quad \& \quad Q = \log(1 + \sqrt{q}\cos p)$$

represent a canonical transformation? If yes, find a generating function.

[7+7=14]

8. (a) Let $f : \mathbb{C} \to \mathbb{C}$ be a function defined by

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z}, & z \neq 0 \\ 0, & z = 0, \end{cases}$$

where $\bar{z}$ denotes the complex conjugate of z. Find the range of z where f is

i. continuous,

ii. differentiable.

(b) i. Find the radius of convergence ρ of the series $\sum_{n=1}^{\infty} f_n(z)$, where

$$f_n(z) = \left(\frac{1}{n^2} + (1+i)^n\right) z^n.$$

ii. Let $\sum_{n=1}^{\infty} g_n(z)$ be any other series having the same radius of convergence ρ. Is it necessarily true that the radius of convergence of $\sum_{n=1}^{\infty}(f_n(z) + g_n(z))$ will be ρ? Justify your answer.

[(3+4)+(4+3)=14]