Part-I

Mathematical and Logical Reasoning

Answer all questions. Each question carries 5 marks.

1. Find the eigenvalues of the matrix $A = (a_{ij})_{n\times n}$, where $a_{ij} = i + j$, *i.e.*,

$$A = \begin{bmatrix} 2 & 3 & 4 & \dots & n+1 \\ 3 & 4 & 5 & \dots & n+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n+1 & n+2 & n+3 & \dots & 2n \end{bmatrix}.$$

[5]

2. Find, with justification, all points of maxima and minima and their respective values of the function $f(x) = x^4 e^{-x^2}$. [5]

3. Consider a particle moving with velocity

$$\frac{dx}{dt} = x + x^2 - 2x^3, \quad x(0) = x_0 \in (0, 1).$$

Prove or disprove that $\lim_{t\to\infty} x(t) \le 1$. [5]

4. A pair of dice is thrown simultaneously until a sum of (the two faces) 4 or 6 appears. Find the probability that a 6 occurs first.

[5]

5. Find the central force which results in the following orbit for a particle:

$$r = a(1 + \cos\theta).$$

[5]

6. A cord passing over a frictionless pulley has a 9 kg mass tied on one end and a 7 kg mass on the other end. Determine the acceleration and the tension of the cord. (Neglect the moment of inertia of the pulley). [5]

Part-II

Physics

Answer any <u>five</u> questions. Each question carries 14 marks.

1. (a) i. Show that the following transformation from canonical variables $\{q, p\}$ to $\{Q, P\}$ is canonical

$$p = m\,\omega\,q\,\cot Q, \quad P = \frac{m\,\omega\,q^2}{2\sin^2 Q}$$

where m and ω are constant parameters.

ii. Express the Hamiltonian $H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$ in terms of Q, P variables.

(b) Kinetic energy (KE) of a relativistic particle, of rest mass m, is given by

$$KE = -mc^2\sqrt{1-\beta^2},$$

where c is the velocity of light in vacuum and $\beta^2 = \mathbf{v^2}/c^2$, with $\mathbf{v}$ being the velocity of the particle. Consider such a particle in a central potential $V(r)$.

i. Write down the Lagrangian L of the particle in spherical polar coordinates $\{r, \theta, \phi\}$.

ii. Identify the cyclic coordinate and derive expression for the conserved momentum. Explain why motion under a central force is restricted to a plane.

iii. Find the Hamiltonian of the system.

[(5 + 2) + (1 + 3 + 3)]

2. (a) Consider a triple pendulum as shown in Fig. 1. Assume that each bob has a mass m, each inextensible string has a length l and each spring has a spring constant k.

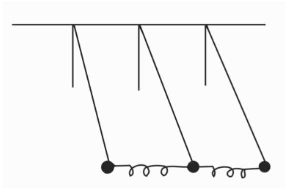

Figure 1: Triple pendulum.

   i. Using the principle of small oscillations, find out the frequencies of oscillation for the system.

   ii. Hence find the normal modes of oscillation and explain the motion in each case with a diagram.

   (b) A train, having a proper length L_0, is approaching a station at a speed v. The light signals emitted by the back-end and front-end of the train reach to an observer on the station simultaneously. Find the distance between the points of emission of the light signals, as measured by the stationary observer.

$[(5+4)+5]$

3. (a) Consider a one-dimensional particle which is confined within the region $0 \leq x \leq a$ and whose wave function is $\psi(x,t) = \sin(\frac{\pi x}{a}) \exp(-i\omega t)$.

   i. Find the potential $V(x)$.

   ii. Calculate the probability of finding the particle in the interval $a/4 \leq x \leq 3a/4$.

   (b) Calculate the commutation relation between the operators

$$\hat{T}_1 = \frac{1}{4}\left(\hat{p}^2 - \hat{x}^2\right) \quad \text{and} \quad \hat{T}_2 = \frac{1}{4}\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right).$$

$[(3+5)+6]$

4. (a) A metal sphere of radius R, having charge q is surrounded by a thick concentric metal shell of inner radius a and outer radius b. The shell carries no net charge.

    i. Find the surface charge density σ at R, a and b.

    ii. Find the potential at the center, using infinity as the reference point.

    iii. Suppose the outer surface is touched to a grounding wire, how would the potential at the center change?

   (b) Consider the scalar potential $V = 0$ and vector potential $\vec{A} = A_o \sin(kx - \omega t)\,\hat{y}$, where A_o, ω, and k are constants.

    i. Find the electric field $\vec{E}$ and magnetic field $\vec{B}$.

    ii. Find the relation between ω and k such that $\vec{E}$ and $\vec{B}$ satisfy Maxwell's equations in vacuum.

$[(3+3+2)+(3+3)]$

5. (a) A rigid quantum rotor with moment of inertia I is governed by the Hamiltonian $\hat{H} = -\frac{\hbar^2}{2I}\frac{d^2}{d\phi^2}$, ϕ being the polar angle in the x-y plane. It has an electric dipole moment p which also lies in the x-y plane. Find the lowest order non-zero shift in the energy of the ground state by using perturbation theory, when a weak electric field $\vec{E}$ is turned on in the x-y plane.

   (b) Consider a charged quantum particle of charge q and mass m moving in the x-y plane in the presence of a constant magnetic field $\vec{B} = B\hat{e}_z$. The Hamiltonian is $\hat{H} = \frac{1}{2m}\left(\hat{\vec{p}} + \frac{q}{c}\hat{\vec{A}}\right)^2$, where $\hat{\vec{p}}$ is the two-dimensional momentum operator and $\hat{\vec{A}}$ is the vector potential corresponding to the magnetic field. Find the energy eigenvalues and the corresponding eigenfunctions of the Hamiltonian.

$[5+(3+6)]$

6. Consider three spin-$\frac{1}{2}$ localized magnetic moments situated on the vertices of an equilateral triangle and having nearest-neighbor ferromagnetic Heisenberg interaction. The Hamiltonian is

$$\hat{H} = -J\left(\hat{\vec{S}}_1.\hat{\vec{S}}_2 + \hat{\vec{S}}_2.\hat{\vec{S}}_3 + \hat{\vec{S}}_3.\hat{\vec{S}}_1\right)$$

with $J > 0$.

(a) What is the dimension of the total Hilbert space of the three localized magnetic moments? Explain with arguments.

(b) Explain, by use of symmetry of the Hamiltonian, the degeneracy structure of the energy eigenspectrum.

(c) The Hamiltonian is easiest to solve by a Fourier transformation of the spin raising and lowering operators. What are the allowed values of the wave-vectors in this case?

(d) Find the energy eigenvalues of all the energy eigenstates, either by Fourier transformation or any other method.

$[3 + 4 + 3 + 4]$

7. A one-dimensional quantum harmonic oscillator (whose ground state energy is $\hbar\omega/2$) is in thermal equilibrium with a heat bath at temperature T.

(a) Find the mean value of the oscillator's energy, $\langle E\rangle$, as a function of T.

(b) What is the value of ΔE, the root-mean-square fluctuation in energy about $\langle E\rangle$?

(c) How do $\langle E\rangle$ and ΔE behave in the limits $kT \ll \hbar\omega$ and $kT \gg \hbar\omega$?

$[3 + 3 + (4 + 4)]$

8. (a) Consider an interacting quantum field theory of two scalar fields ϕ and ψ in $(3+1)$-dimensions with the interaction term of the form $\lambda\phi\psi^3$, λ being the coupling constant.

   i. Write down the action of the theory.

   ii. Draw possible Feynman diagrams for any two of the following processes, to the lowest order of interaction:
   • one ϕ particle and one ψ particle going to two ψ particles,
   • two ϕ particles going to four ψ particles,
   • two ϕ particles going to two ψ particles.
   In each case label the momenta in the propagators for some arbitrary initial and final momentum configurations.

   (b) The mean lifetime of a charged π meson (commonly known as pions) at rest is 2.6×10^{-8} sec. A mono-energetic beam of high-energy pions, produced by an accelerator, travels a distance of 10 meters. In the process 10% of the pions decay.

   i. Find out the momentum of the pions.

   ii. Also find their kinetic energy.

$[(2 + (3 + 3)) + (3 + 3)]$

Part-III

Applied Mathematics

Answer any <u>five</u> questions. Each question carries 14 *marks.*

- The symbol $\mathbb{R}$ will denote the set of all real numbers.
- The symbol $\mathbb{C}$ will denote the set of all complex numbers.
- The symbol O will denote the null matrix.
- The symbol $\mathrm{Tr}(A)$ will denote the trace of the matrix A.
- The symbol $A \subset B$ will denote "A is a proper subset of B".

1. (a) Let G be a group generated by a and b subject to the relations $aba = b^3$ and $b^5 = 1$, where 1 is the identity element of G.

    i. Show that G is abelian.

    ii. Give an example of a non-abelian group with two generators.

   (b) Let $\mathcal{R}$ be a ring with the multiplicative identity 1. Suppose that the order of $\mathcal{R}$ is p^2 for some prime number p. Then prove that $\mathcal{R}$ is a commutative ring.

   $[(6 + 2) + 6]$

2. (a) Let E be a Lebesgue measurable subset of $\mathbb{R}$ with $\mu_L(E) = 1$. Show that there exists a Lebesgue measurable set $A \subset E$ such that $\mu_L(A) = \frac{1}{2}$.

   (b) Show that a metric space X is connected if and only if for every non-empty proper subset $A \subset X$, the boundary ∂A is non-empty.

   $[7 + 7]$

3. (a) i. Derive the general harmonic function of the form $u(x, y) = \phi(x^2 + y^2)$, where $x, y \in \mathbb{R}$ and ϕ is a real valued function.

ii. Explain whether analytic functions of the form $f(z) = u(x, y) + iv(x, y)$ with $u(x, y) = \phi(x^2 + y^2)$ exist and if they exist, find them.

(b) Find all the singularities and their types of the complex valued function

$$f(z) = \frac{1}{(z^4 + z^3 + z^2 + z + 1)(z - 3)}, \quad z \in \mathbb{C},$$

and evaluate $\int_{|z|=2} f(z)dz$.

[(3 + 4) + (3 + 4)]

4. (a) i. Find all the entire functions f such that $|f(z)| = 1$ for $|z| = 1$.

ii. Is their any non-constant entire function f such that

$$f(z + 1) = f(z) = f(z + i), \forall z \in \mathbb{C}?$$

Justify your answer.

(b) i. Find the rate of change of

$$f(x, y) = 1 - \frac{x^2}{4} - \frac{y^2}{4}$$

at the point $P(1, 0)$ in the direction of unit vector 45° to the x-axis.

ii. Also find a direction at P along which $f(x, y)$ is not changing.

[(3 + 4) + (4 + 3)]

5. (a) Find the extremum of the function

$f(x, y) = x^2+2y^2-x$, subject to the constraint $x^2+y^2 \leq 1$.

(b) Find the integral surface of the partial differential equation

$$(x-y)\frac{\partial z}{\partial x} + (y-x-z)\frac{\partial z}{\partial y} = z$$

which passes through the curve $z = 1, x^2 + y^2 = 1$.

[7 + 7]

6. (a) i. Find the range of A and B such that the following transformation

$$Q = -p, \quad P = Ap^2 + Bq$$

becomes a canonical transformation.

ii. Let

$$H = \frac{p^2}{2m} + mgq$$

be the Hamiltonian for a particle of mass m moving vertically in a uniform gravitational field g. Find the new Hamiltonian for the new canonical variables (Q, P) mentioned in (i) and determine the condition to make Q as cyclic coordinate in the new Hamiltonian.

(b) The velocity potential function for a two-dimensional flow is given by $\phi = x^2 - y^2$,

i. determine the velocity components in x- and y-direction,

ii. show that the velocity components satisfy the conditions of flow continuity and irrotationality,

iii. determine stream function and flow rate between streamlines $(2, 0)$ and $(2, 2)$, and

iv. show that the streamlines and potential lines intersect orthogonally at the point $(2, 2)$.

[(3 + 4) + (2 + 2 + 2 + 1)]

7. (a) i. Show that the trace of product of two Hermitian matrices is a real number.

   ii. Hence, or otherwise, show that

$$\mathrm{Tr}\left(A^2\right) = \mathrm{Tr}(A^*A) \Rightarrow A^2 = A^*A,$$

   where A^* denotes the Hermitian adjoint (i.e., conjugate transpose) of A.

   (b) Let A, B are complex matrices of order 2×2 such that $A = AB - BA$. Show that $A^2 =$O.

[(4 + 5) + 5]

8. (a) Consider the following system for $x > 0$ and bifurcation parameter $r > 0$:

$$\frac{dx}{dt} = 3\ln x - rx^2.$$

   i. A bifurcation occurs at a critical bifurcation value r_c. Sketch phase portraits for $r < r_c, r = r_c$, and $r > r_c$.

   ii. Sketch the bifurcation diagram.

   iii. What type of bifurcation occurs? Explain.

   iv. Calculate r_c (tough, probably do last).

   (b) Consider the differential equation $\frac{dy}{dx} = \frac{2y}{x}$ with initial condition $y(x_0) = y_0$. Discuss the existence and unique solution of this differential equation.

[(3 + 2 + 2 + 2) + 5]