Sample questions – MTB

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1. $\mathbb{N}$ denotes the set of all positive integers.
2. $\mathbb{Z}$ denotes the set of all integers.
3. $\mathbb{Q}$ denotes the set of all rational numbers.
4. $\mathbb{R}$ denotes the set of all real numbers.
5. $\mathbb{C}$ denotes the set of all complex numbers.

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1. Let A, B be linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $m \in \mathbb{N}$ such that $0 \neq A \neq \mathrm{id}_{\mathbb{R}^n}$, $A = A^2$, $AB = BA$ and $B^m = 0$. Show that $B^{n-1} = 0$.

2. Find the number of distinct group homomorphisms from $\mathbb{Z}_6$ to the symmetric group S_5.

3. For $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, define

$$S_{\underline{a}} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} a_j x_j^2 = 1 \right\}.$$

   Let $F = \{\underline{a} \in \mathbb{R}^n : S_{\underline{a}} \text{ is compact }\}$. Identify the set F.

4. Let α be a real root of $x^3 - 3x - 1 \in \mathbb{Q}[x]$.

   (a) Show that $\mathbb{Q}(\alpha)$ is a degree 3 extension of $\mathbb{Q}$.

   (b) Express $\alpha^4 + 2\alpha^3 + 3 \in \mathbb{Q}(\alpha)$ in the form of $a + b\alpha + c\alpha^2$ where $a, b, c \in \mathbb{Q}$.

   (c) Express $(\alpha - 1)^{-1} \in \mathbb{Q}(\alpha)$ in the form of $a + b\alpha + c\alpha^2$ where $a, b, c \in \mathbb{Q}$.

5. Let R denote the ring $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$.

   (a) Show that R is isomorphic to the ring $\frac{\mathbb{Z}[x]}{(x^2-5)}$.

   (b) Find all the prime ideals of R containing 6 .

6. Let R denote the ring $\frac{\mathbb{Z}[x,y]}{(2,x^2,y^2)}$. Prove that every element of R is either a unit or a nilpotent.

7. Let V be a finite dimensional complex inner product space and $T : V \to V$ is linear.

   (a) Prove that T is normal (that is, $T \circ T^* = T^* \circ T$) if and only if $\|Tx\| = \|T^*x\|$ for all $x \in V$.

   (b) If T is normal, then show that there exists a linear map $U : V \to V$ satisfying $U \circ U^* = \mathrm{id}_V = U^* \circ U$ and $U \circ T = T^*$.

8. Consider $\mathbb{C}^n$ as a vector space over $\mathbb{C}$ where $n > 1$. Suppose W_1, W_2 are two $(n-1)$-dimensional complex subspaces of $\mathbb{C}^n$ such that $W_1 \neq W_2$. If $\mathbb{C}^n$ has the topology given by the metric

$$d\left((z_1, \ldots, z_n), (w_1, \ldots, w_n)\right) = \left[\sum_{j=1}^{n} |z_j - w_j|^2\right]^{\frac{1}{2}}$$

for $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{C}^n$, then show that $\mathbb{C}^n \setminus (W_1 \cup W_2)$ is path connected.

9. Let $X_i, i \geq 1$ be i.i.d. discrete random variables with mean μ and variance σ^2. Let $k > 1$. Define the sequence

$$Y_n := \frac{X_1 X_2 \ldots X_k + X_2 X_3 \ldots X_{k+1} + \ldots + X_{n-k+1} X_{n-k+2} \ldots X_n}{n}.$$

Find $\lim_{n\to\infty} \mathbb{E}[Y_n]$ and $\lim_{n\to\infty} n^{3/4} \mathbb{E}\left[(Y_n - \mathbb{E}[Y_n])^2\right]$ where $\mathbb{E}$ denotes the expectation.

10. Let X_n = number of heads obtained from n independent coin tosses with probability of head p. Let p_n be the probability that X_n is an even number.

    (a) Show that $p_{n+1} = (1-2p)p_n + p$.

    (b) Show that $\lim_{n\to\infty} p_n$ exists and find the limit.