$\mathbb{Z} =$ the set of integers

$\mathbb{N} = \{n \in \mathbb{Z} : n \geq 1\}$

$\mathbb{R} =$ the set of real numbers

$\mathbb{Q} =$ the set of rational numbers

$\mathbb{C} =$ the set of complex numbers

1. Let $X = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\} \subset \mathbb{R}^2$. On X consider the equivalence relation given by

$$(x, 0) \sim (x, 1), \quad \text{if} \quad x \neq 0,$$
$$(0, 0) \sim (0, 0),$$
$$(0, 1) \sim (0, 1).$$

Define $Y := X/\sim$, the set of equivalence classes under $\sim$ with the quotient topology, and let

$$\Delta = \{(y, y) | y \in Y\} \subset Y \times Y.$$

Prove that Δ is not closed in $Y \times Y$, where $Y \times Y$ has the product topology.

2. Let $X \subset \mathbb{R}^2$ be the set of all points having at least one coordinate irrational. Is X path connected? Explain your answer.

3. Let A and B be two 9×9 complex matrices such that

$$AB = BA^3.$$

Suppose none of the eigenvalues of A has modulus 1. Show that A and B have a common eigenvector.

4. Let ω be a bilinear form on $\mathbb{R}^m$ that satisfies the following properties:

   (a) $\omega(x, x) = 0$ for all $x \in \mathbb{R}^m$, and

   (b) $\omega(x, y) = 0$ for all $y \in \mathbb{R}^m$ implies $x = 0$.

   Prove that m is an even integer and there exists a non-singular, skew-symmetric matrix K such that

   $$\omega(x, y) = x^T K y,$$

   for all $x, y \in \mathbb{R}^m$.

5. Find the number of ordered 4-tuples (a, b, c, d), where a, b, c, d are odd natural numbers, such that

   $$a + b + c + d = 50.$$

6. Consider the sequence of integers $\{1, 11, 111, 1111, \cdots\}$ whose nth term is given by

   $$a_n = 1 + 10 + \cdots + 10^{n-2} + 10^{n-1}.$$

   Let $p > 5$ be a prime number. Show that there are infinitely many n such that p divides a_n.

7. Let G be a finite group with the property that given any two non-trivial elements $x, y \in G$, there exists an automorphism ϕ of G such that

   $$\phi(x) = y.$$

   Prove that G is abelian.

8. Let $n \geq 2$ be a natural number. Consider the polynomial

   $$f(X) = 1 + X + X^2 + \cdots + X^{n-1} \in \mathbb{Z}[X].$$

   Suppose $f(X)$ is irreducible in $\mathbb{Z}[X]$. Show that n is a prime number.