- **The symbol $\mathbb{Z}$ will denote the set of all integers.**
- **The symbol $\mathbb{R}$ will denote the set of all real numbers.**
- **The symbol $\mathbb{Q}$ will denote the set of all rational numbers.**
- **The symbol $\mathbb{C}$ will denote the set of all complex numbers.**
- **For $n \in \{1, 2, \cdots\}$, $M_n(\mathbb{R})$ will denote the set of all $n \times n$ matrices with real entries.**
- **The symbol I_n will stand for the $n \times n$ identity matrix.**
- **Any ring R considered below is assumed to contain a multiplicative identity 1_R.**
- **A ring homomorphism $f : R \to S$ is always assumed to satisfy $f(1_R) = 1_S$.**

1. Let A be an element of $M_3(\mathbb{R})$. Assume that there are linearly independent vectors v_1, v_2, v_3 in $\mathbb{R}^3$ and real numbers $\lambda_1, \lambda_2, \lambda_3$ such that for all $i = 1, 2, 3$,

$$A(v_i) = \lambda_i v_i.$$

Let

$$W = \{B \in M_3(\mathbb{R}) : BA = AB\}.$$

Compute the dimension of the vector space W in the following two cases:

(a) when $\lambda_1 = \lambda_2 = \lambda_3$.

(b) when $\lambda_1 = \lambda_2$ and $\lambda_3 \neq \lambda_1$.

2. Let $J : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal linear transformation such that $J^2 = -I_n$.

   (a) Prove that n is even.

   (b) Prove that if $v \in \mathbb{R}^n$, then v is orthogonal to $J(v)$.

   (c) Assume that $n = 4$. Show that there is an orthonormal basis of $\mathbb{R}^4$ such that the matrix of J with respect to this basis is

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3. Let p and q be distinct prime numbers.

   If G is a group of order pq such that G has at least two conjugacy classes of size 1, then prove that G is cyclic.

4. Consider the rings

$$R_1 = \frac{\mathbb{Q}[x]}{(x^2)}, \quad R_2 = \frac{\mathbb{Q}[x]}{(x^2-1)}, \quad R_3 = \frac{\mathbb{Q}[x]}{(x^2-2)}.$$

   Prove or disprove each of the following statements:

   (a) R_1 is isomorphic to R_2 as rings,

   (b) R_2 is isomorphic to R_3 as rings,

   (c) R_1 is isomorphic to R_3 as rings.

5. Consider the ring

$$R = \mathbb{Z}[\sqrt{-1}] \cong \frac{\mathbb{Z}[x]}{(x^2+1)}.$$

   Let F be a finite field with 25 elements. Prove that there does not exist any surjective ring homomorphism from R to F.

6. Let G be a group acting on a ring R which satisfies the equations:

$$g.(x+y) = g.x + g.y,\ \ g.(xy) = (g.x)(g.y)$$

for all $x, y \in R$ and for all $g \in G$.

(a) Prove that

$$R^G := \{r \in R : g.r = r\ \forall\ g \in G\}$$

is a subring of R.

(b) In the above notation, let $R = \mathbb{C}[x, y]$ and

$$G = C_3 = \{g : g^3 = e\}$$

be the cyclic group of order 3.

Assume that the action of G on R satisfies the following equations:

$$g.x = \omega x,\ \ g.y = \omega^2 y,$$

where $\omega = \cos(\frac{2\pi}{3}) + \sqrt{-1}\sin(\frac{2\pi}{3})$ is a cube root of unity.

Observe that $x^3, y^3 \in R^G$.

Prove that $\frac{R^G}{(x^3,y^3)}$ is a vector space over $\mathbb{C}$ and compute its dimension.

7. Let $X \subseteq \mathbb{R}^2$ be defined as

$$X = \Big(\{0\} \times [\tfrac{1}{2}, 1]\Big) \bigcup \Big(\bigcup_{n\geq 1} (\{\tfrac{1}{n}\} \times [0, 1])\Big) \bigcup \Big([0, 1] \times \{0\}\Big).$$

X is equipped with the subspace topology from $\mathbb{R}^2$.

Prove or disprove the following statement:

X is connected.

8. Let X be a compact metric space having at least two elements and let $x_0 \in X$. Suppose that $X \setminus \{x_0\}$ is connected.

   Consider the following statements:

   (i) X is connected.

   (ii) $X \setminus \{x_0\}$ is compact.

   (a) Prove that either (i) or (ii) is true.

   (b) Prove that (i) and (ii) cannot both be true.