

- \mathbb{R} denotes the set of real numbers.
- \mathbb{Q} denotes the set of rational numbers.
- \mathbb{Z} denotes the set of integers.

Q 1. Let f be a measurable function on \mathbb{R} such that $\int_I f d\lambda = 0$ for all bounded intervals $I \subset \mathbb{R}$, where λ is the Lebesgue measure on \mathbb{R} . Show that

$$\lambda(\{x \in \mathbb{R} : f(x) \neq 0\}) = 0.$$

Q 2. Let $C_{\mathbb{R}}[0, 1]$ denote the Banach space of all continuous real-valued functions on $[0, 1]$ with norm

$$\|f\|_{\infty} = \sup\{|f(x)| : x \in [0, 1]\}.$$

Show that there is no inner product on $C_{\mathbb{R}}[0, 1]$ that induces the norm $\|\cdot\|_{\infty}$.

Q 3. Find the smallest positive integer n such that

$$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

is isomorphic to a subgroup of S_n , where S_n is the group of all permutations of $\{1, 2, \dots, n\}$.

Q 4. Let $p = 1 + 4n$ be a prime in \mathbb{Z} .

(a) Show that $(2n)!$ is a solution to the congruence

$$x^2 \equiv -1 \pmod{p}.$$

(b) Use (a) to show that p is not a prime element in $\mathbb{Z}[i]$.

(c) Use (b) to show that $p = a^2 + b^2$ for some $a, b \in \mathbb{Z}$.

Q 5. Fix a $p \in [1, \infty]$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $\{a_n x_n\}_{n=1}^{\infty} \in \ell^p$ for all sequences $\{x_n\}_{n=1}^{\infty} \in \ell^p$. Show that $\{a_n\}_{n=1}^{\infty} \in \ell^{\infty}$.

Q 6. Let $\{B_{\alpha} : \alpha \in A\}$ be a family of pairwise disjoint Lebesgue measurable subsets of $[0, 1]$ each of positive measure. Show that the family is countable.

- Q 7.** Let $C[0, 1]$ denote the Banach space of all continuous complex-valued functions on $[0, 1]$ with norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in [0, 1]\}.$$

Let $g : [0, 1] \rightarrow [0, 1]$ be a non-constant continuous function. Consider the operator $T : C[0, 1] \rightarrow C[0, 1]$ defined by $Tf = f \circ g$, that is, $Tf(x) = f(g(x))$, $f \in C[0, 1]$, $x \in [0, 1]$.

Show that the image of the unit ball in $C[0, 1]$ under T is not compact in $C[0, 1]$.

- Q 8.** Let p be a prime and $K = \mathbb{Q}(\alpha)$, where $\alpha^3 = p$. Find the minimal polynomial of $\alpha + \alpha^2$ over \mathbb{Q} . Justify your answer completely.

- Q 9.** Let $n > 1$ be an odd integer. Show that n does not divide $3^n + 1$.

- Q 10.** (a) Let R be a commutative ring with unity. Let $a, b, c \in R$ be such that there exist $x, y, z \in R$ with

$$xa + yb + zc = 1.$$

Show that there exist $\alpha, \beta, \gamma \in R$ such that

$$\alpha a^{15} + \beta b^{16} + \gamma c^{17} = 1.$$

- (b) Let R be a commutative ring with unity such that for each $a \in R$ there exists an integer $n(a) > 1$ such that $a^{n(a)} = a$. Prove that every prime ideal of R is maximal.