

Sample questions – MTA

1. \mathbb{N} denotes the set of all positive integers.
 2. \mathbb{Z} denotes the set of all integers.
 3. \mathbb{Q} denotes the set of all rational numbers.
 4. \mathbb{R} denotes the set of all real numbers.
 5. \mathbb{C} denotes the set of all complex numbers.
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1. Show that $\lim_{N \rightarrow \infty} p(N) \int_N^\infty e^{-x^2} dx = 0$ for all polynomial $p(x)$ in $\mathbb{R}[x]$.
2. Let \mathcal{H} and \mathcal{K} be complex Hilbert spaces. Suppose $\{T_n\}_{n \in \mathbb{N}}$ is a sequence of bounded linear operators from \mathcal{H} to \mathcal{K} satisfying the following condition: For every $\xi \in \mathcal{H}$ and for every $\eta \in \mathcal{K}$, the sequence $\{\langle T_n \xi, \eta \rangle_{\mathcal{K}}\}_{n \in \mathbb{N}}$ of complex numbers is Cauchy.

(a) how that $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$.

(b) Show that there exists a bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{K}$ such that $\langle T \xi, \eta \rangle_{\mathcal{K}} = \lim_{n \rightarrow \infty} \langle T_n \xi, \eta \rangle_{\mathcal{K}}$ for every $\xi \in \mathcal{H}, \eta \in \mathcal{K}$.

3. Let Ω be an open subset of \mathbb{C} . Suppose f is a meromorphic function on Ω having exactly one zero at z_0 and exactly one pole at z_1 where m is the order of zero at z_0 and n is the order of the pole at z_1 . If $g : \Omega \rightarrow \mathbb{C}$ is analytic and D is an open disc such that $z_0, z_1 \in D$ and $\bar{D} \subset \Omega$, then show that

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} g(z) dz = mg(z_0) - ng(z_1)$$

where ∂D is the boundary of D .

4. Let $z_1, \dots, z_n \in \mathbb{C}$ such that $|z_j| = 1$ for all $j = 1, \dots, n$. Consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$f(w) = (w - z_1) \cdots (w - z_n) \text{ for all } w \in \mathbb{C}.$$

Prove that the set $\{|f(w)| : |w| = 1\}$ contains the interval $[0, 1]$.

5. Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of Borel measurable subsets of $[0, 1]$ satisfying $\lambda(A_n) = 1$ for all n (where λ denotes the Lebesgue measure on $[0, 1]$). Prove that $\bigcap_{n \in \mathbb{N}} A_n$ is dense in $[0, 1]$.

6. Let $p \in [1, \infty)$. Consider the vector space

$$\ell^p = \left\{ f : \mathbb{N} \rightarrow \mathbb{C} : \sum_{k \in \mathbb{N}} |f(k)|^p < \infty \right\}$$

with the usual norm $\|f\|_p = \left[\sum_{k \in \mathbb{N}} |f(k)|^p \right]^{\frac{1}{p}}$ for $f \in \ell^p$. Suppose $\|\cdot\|$ is another norm on ℓ^p with the following properties:

- (a) $(\ell^p, \|\cdot\|)$ is complete;
- (b) If a sequence $\{f_n\}_n$ converges to f in $(\ell^p, \|\cdot\|)$, then $\lim_{n \rightarrow \infty} f_n(k) = f(k)$ for all $k \in \mathbb{N}$.

Prove that $\|\cdot\|$ is equivalent to $\|\cdot\|_p$.

7. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of continuous functions from \mathbb{R}^d to \mathbb{R} , $d \geq 1$. Suppose there exist $M_1, M_\infty > 0$ and $c \in \mathbb{R}$ satisfying the following conditions:

- (a) $\|f_n\|_1 \leq M_1$ and $\|f_n\|_\infty \leq M_\infty$ for all $n \in \mathbb{N}$, and
- (b) $\lim_{n \rightarrow \infty} f_n(x) = c$ for all $x \in \mathbb{R}^d$

Prove that $c = 0$. (Here $\|g\|_1 = \int_{\mathbb{R}^d} |g(x)| dx$ and $\|g\|_\infty = \sup_{x \in \mathbb{R}^d} |g(x)|$.)

8. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function which is not a polynomial. Suppose $\{P_n\}_{n \geq 1}$ is a sequence of polynomials (with real coefficients) that converges uniformly to the function f on $[0, 1]$. Prove that $\{\text{degree}(P_n) : n = 1, 2, \dots\}$ is unbounded.